Alternating Layered Ising Models : Effects of connectivity and proximity

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Experiments on ^{4}He by Gasparini an Coworkers

Experiments on ^{4}He at the superfluid transition

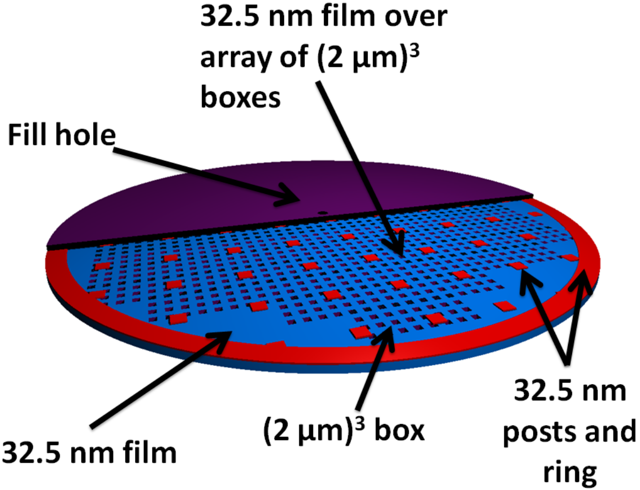


Figure : Experiments on small boxes of helium were coupled through a thin helium film.

Model

Alternating Layered Ising Model:

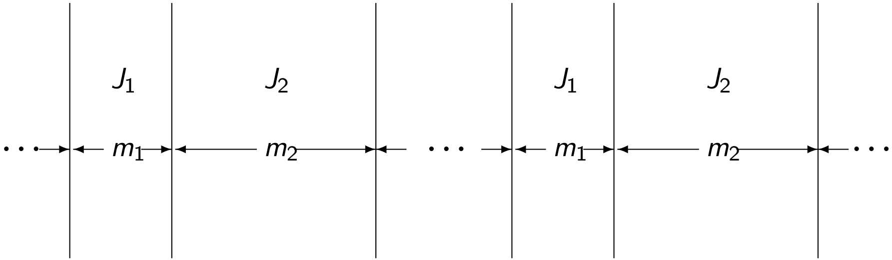


Figure : The model consists of infinite strips of width m_1 in which the coupling energy between the nearest neighbor Ising spins is J_1 separated by other infinite strips of width m_2 whose coupling J_2 is "weaker". $(\sigma = \pm 1)$.

Relative strengths r and relative separations s

$$r = J_2/J_1 < 1, \qquad s = m_2/m_1.$$

$r = 0$ $(J_2 = 0)$: 1-D Ising: No discontinuities

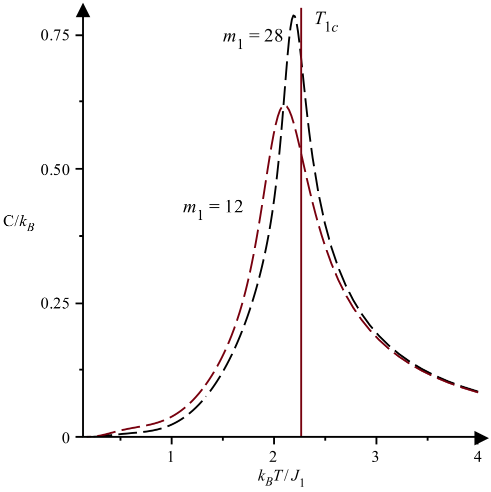


Figure : Specific heats for $r = 0$: non-interacting infinite strips of finite width m_1.

- $J_2 = 0$, the model $\rightarrow$ 1D. Specific heat not divergent, but rather has a fully analytic rounded peak.
- The temperature of the maximum T_{1max} is below the bulk critical point T_{1c} and increases as m_1 increases; it approaches T_{1c} as $m_1 \rightarrow \infty$.
- Finite-size scaling holds.

$r \neq 0$: 2-D Ising: $\alpha = 0$, $\beta = 1/8$, $\nu = 1$.

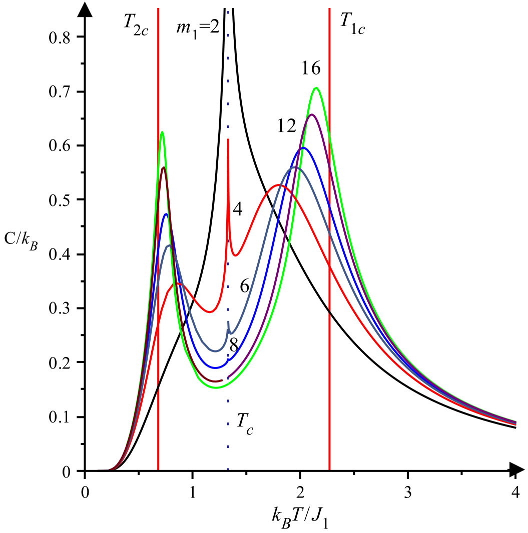


Figure : Specific heats for $r = 0.3$ and $s = 1$ for $m_1 = m_2 = 2, 4, 6, 8, 12$ and 16. Dotted vertical line : T_c.

- Specific heats divergent at T_c logarithmically.
- m_1 increases, the divergence becomes a barely visible spike.
- and two rounded peaks appear and move toward the limiting values T_{1c} and T_{2c} as $m_1 = m_2$ increases.

Critical Temperature $T_c(r, s)$

Critical temperature for random layered models [McCoy and Wu, Fisher]

$$2\langle\langle J_y \rangle\rangle = k_B T_c \langle\langle \ln \coth(J_x / k_B T_c) \rangle\rangle,$$

where the brackets $\langle\langle \cdot \rangle\rangle$ denote an average over the distribution,

Critical temperature for alternating layered models

$$2J_1 m_1 + 2J_2 m_2 = k_B T_c [m_1 \ln \coth(J_1 / k_B T_c) + m_2 \ln \coth(J_2 / k_B T_c)].$$

The critical temperature : $T_c = T_c(r, s)$

$$2J_1(1 + rs) = k_B T_c [\ln \coth(J_1 / k_B T_c) + s \ln \coth(r J_1 / k_B T_c)],$$

Specific Heat

$$C(T; m_1, m_2; J_1, J_2) \simeq A(r, s) \ln[T / T_c(r, s) - 1]$$

Layered Ising model

Model

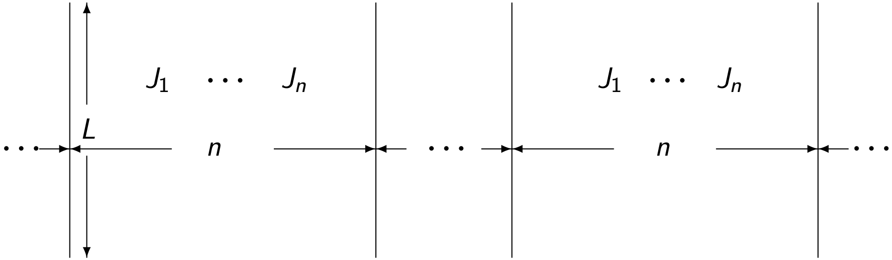


Figure : The model consists of strips of length L and width n in which the coupling energy between the nearest neighbor Ising spins is J_k for $k = 1, \cdots, n$, in the limit $L \to \infty$

Critical Point in the Layered Ising Models

Pffaffian method: Cyclic boundary — vertical direction; open boundary — horizontal direction. As $L \to \infty$, the free energy becomes an integral over $\theta = 2\pi/L$. Since if L finite, no singularity!!! The integrand is singular only at $\theta = 0$.

Amplitude

Expanding about $\theta \sim 0$:

$$\mathcal{I}_s = A_1^2 (J_1/k_B)^2 [(1/T) - (1/T_c)]^2 + A_2^2 \theta^2 + ...,$$

Alternating Layered Ising model

$$A(r,s) \approx \frac{16 K_{1c}^2 sq}{\pi(s+1)\sinh[2sq/(1+s)]}, \quad q = 2K_{1c}(1-r)m_1$$

Amplitue $A(r,s) \to 0$ exponentially as $m_1 \to \infty$.

Fibonacci Ising Models by Tracy 1988

$$S_{n+1} = S_n S_{n-1}, \quad S_0 = B, \quad S_1 = A, \quad S_2 = AB, \quad S_3 = ABA,$$
$$S_4 = ABAAB, \quad S_5 = ABAABABA, \quad S_\infty = \lim_{n\to\infty} S_n.$$

He shows that the amplitude is finite in the limit $n \to \infty$.

$m_2 = 2m_1$, $s = 2$

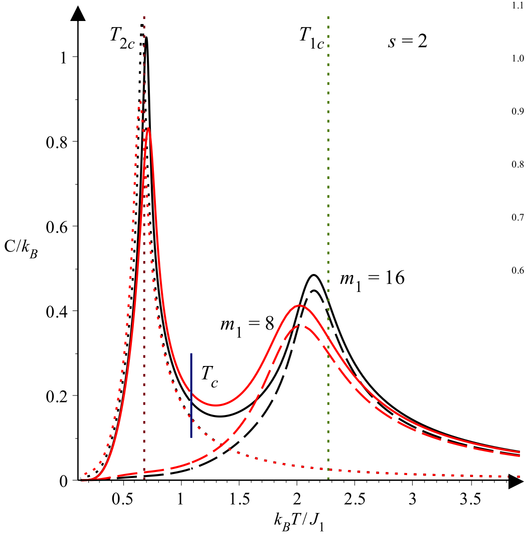
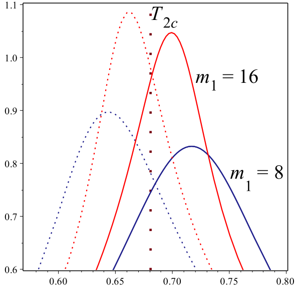


Figure : Specific heats for $r = 0.3$ and $s = 2$. The dashed plots: $J_2 = 0$; dotted lines: $J_1 = 0$. ▸ Go to uppersc



- Upper peaks below T_{1c}. Solid curve above dash.
- Lower peaks above T_{2c}, differ from dotted lines
- In agreement with the experiments of Gasparini.

$m_1 = m_2 = 16,\ r = 0.5, 0.7, 0.9$

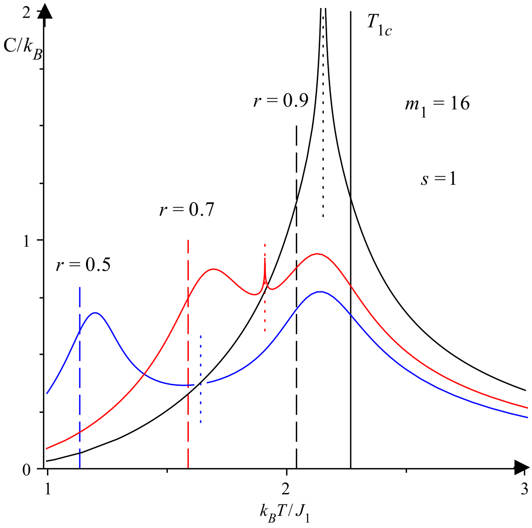


Figure : Specific heats for $r = 0.5, 0.7, 0.9$ and $m_1 = 16$; $s = 1$.

- T_{2c} and T_c increases as r increases.
- Logarithmic divergence is visible for $r = 0.7$,
- and dominates entirely for $r = 0.9$.

Scaling behavior near T_{1c}

Goto s2

$$C_1(J_1, J_2; T) = (1+s)[C(J_1, J_2; T) - C(0, J_2; T)]$$

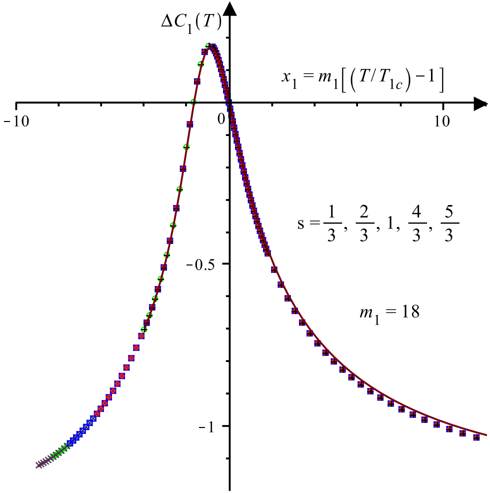


Figure : Plots of $\Delta C_1(J_1, J_2; T)$ (solid minus dotted, and subtract its value at T_{1c}).

Data collapse: $\Delta C_1(T) = C_1(T) - C_1(T_{1c})$ are independent of m_2.

The solid curve is the plot of the specific heat of an infinite strip of width $m_1 = 18$ and coupling J_1 when its value at T_{1c} is subtracted.

Scaling behavior of Alternating Layered Model:

- When $T \sim T_{1c}$, $\xi_1(T) = 1/|t_1| \gg 1$, $\xi_2(T)$ small, (ξ_i are the bulk correlation length of coupling J_i). When $m_2/\xi_2(T) \gg 1$,

$$C_1(J_1, J_2; T) = \frac{m_1 + m_2}{m_1}[C(J_1, J_2; T) - C(0, J_2; T)]$$

is independent of m_2.

- In the scaling limit: $t_1 \to 0$, $m_1 \to \infty$: $x_1 = t_1 m_1$

$$\Delta C_1(J_1, J_2; T) = C_1(J_1, J_2; T) - C_1(J_1, J_2; T_{1c})$$
$$\approx C^{strip}(J_1; m_1; T) - C^{strip}(J_1; m_1; T_{1c}) \approx Q(x_1) - Q(0).$$

- Similarly $T \sim T_{2c}$, $\xi_2(T) = 1/|t_2| \gg 1$, $\xi_1(T)$ small, so that when $m_1/\xi_1(T) \gg 1$

$$C_2(J_1, J_2; T) = \frac{m_1 + m_2}{m_2}[C(J_1, J_2; T) - C(J_1, 0; T)],$$

is independent of m_1.

- In the scaling limit $t_2 \to 0$, $m_2 \to \infty$ with fixed $x_2 = t_2 m_2$,

$$\Delta C_2(J_1, J_2; T) = C_2(J_1, J_2; T) - C_2(J_1, J_2; T_{2c})$$
$$\approx Q(-x_2) - Q(0).$$

Plots of Scaling behavior near T_{2c}

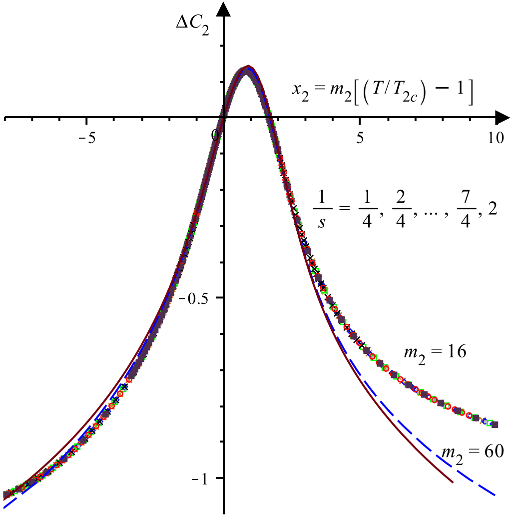


Figure : Plots of $\Delta C_2(J_1, J_2; T)$ for $m_2 = 16$, and $m_1 = 4, 8, \cdots, 32$.

Data collapse: $\Delta C_2(T) = C_2(T) - C_2(T_{2c})$ are independent of m_1.

The solid curve is scaling function $Q(-x_2) - Q(0)$, and dashed line for $m_2 = 60$.

Free Energy $f_s(J_1, J_2; T)$

$$f_s(J_1, J_2; T) = \frac{1}{m_1 + m_2} \int_0^{\frac{1}{2}\pi} \frac{\mathrm{d}\theta}{\pi} \ln \tfrac{1}{2}\left[W + \sqrt{W^2 - 4}\right],$$

$$W = U_1^+ U_2^+ + U_1^- U_2^- + \tfrac{1}{2}(C_1 C_2 - 1) V_1 V_2,$$

The terms $U_i^+ = U^+(t_i, m_i)$ are related to the free energy $f^\infty(m_i; J_i; T)$ of an infinite strip of width m_i with coupling J_i in which we have introduced the basic temperature variables, t_i, via

$$t_i \approx 2K_{ic} - 2K_i \approx 2K_{ic}(T/T_{ic} - 1), \quad 2K_{ic} = \ln(\sqrt{2} + 1),$$

$$f_s^\infty(m_i; J_i; T) = \frac{1}{m_i} \int_0^{\frac{1}{2}\pi} \frac{\mathrm{d}\theta}{\pi} \ln U^+(t_i, m_i).$$

The remaining terms are related to the interaction between the strips. If $J_2 \to 0$, so that the system becomes uncoupled, we find $U_2^- = 0$ and $V_2 = 0$.

Scaling behavior of a single infinite strip of width m_1:

$$U_i^+ = U^+(t_i, m_i) = \tfrac{1}{2}(\alpha_i^{m_i} + \alpha_i^{-m_i}) + \tfrac{1}{2}(\alpha_i^{m_i} - \alpha_i^{-m_i})g_i,$$

$$\alpha_i^{\pm 1} = \mathfrak{c}_i \pm 2Y_i, \quad \mathfrak{c}_i = 2t_i^2 + 2\omega^2 + 1, \quad g_i = h_i/Y_i,$$
$$Y_i = \tfrac{1}{2}\sqrt{\mathfrak{c}_i^2 - 1} = \sqrt{(t_i^2 + \omega^2)(t_i^2 + \omega^2 + 1)},$$

$$\alpha_i^{m_i} = \mathrm{e}^{2m_i \,\mathrm{arcsin}\, \sqrt{t_i^2+\omega^2}} \gg \alpha_i^{-m_i} \quad \text{if} \quad m_i|t_i| >> 1.$$

- When $m_1/\xi_1(T) >> 1$, the system behaves as 2D Ising;

  $U^+(t_1, m_1) = \alpha_1^{m_1}\tfrac{1}{2}(1 + g_1)$
  $C^{strip}(J_1; m_1; T) = \text{Bulk specific heat} + (1/m_1)\text{Surface energy}.$

- $\alpha = 0$ and $\nu = 1$: Near T_{1c}, it was shown that finite size scaling holds,

  $$C^{strip}(J_1; m_1; T) = A_0 \ln m_1 + Q(x_1) + O(m_1^{-1}, m_1^{-1} \ln m_1),$$
  $$x_1 = m_1 t_1 \propto m_1/\xi_1(T).$$

Behavior of the coupled system near T_{1c}

$$f_s(J_1, J_2; T) = \frac{1}{m_1 + m_2} \int_0^{\frac{1}{2}\pi} \frac{\mathrm{d}\theta}{\pi} \ln \tfrac{1}{2}\left[W + \sqrt{W^2 - 4}\right],$$

where

$$\begin{aligned} W \;=\; & \tfrac{1}{2}(\alpha_1^{m_1} + \alpha_1^{-m_1})(\alpha_2^{m_2} + \alpha_2^{-m_2}) + \\ & \tfrac{1}{2}(\alpha_1^{m_1} - \alpha_1^{-m_1})(\alpha_2^{m_2} - \alpha_2^{-m_2})G(t_1, t_2; \omega), \end{aligned}$$

$$f_s(0, J_2; T) = \frac{1}{m_1 + m_2} \int_0^{\frac{1}{2}\pi} \frac{\mathrm{d}\theta}{\pi} \ln U^+(t_2, m_2),$$

$T \sim T_{1c}$: $\xi_2(T)$ small. When $m_2/\xi_2(T) >> 1$, drop $\alpha_2^{-m_2}$

$$W = \alpha_2^{m_2}\,\tfrac{1}{2}\mathcal{I}_1, \qquad U^+(t_2, m_2) = \alpha_2^{m_2}\,\tfrac{1}{2}(1 + g_2),$$
$$\mathcal{I}_1 = [(\alpha_1^{m_1} + \alpha_1^{-m_1}) + (\alpha_1^{m_1} - \alpha_1^{-m_1})G(t_1, t_2; \omega)].$$

$C_1(J_1, J_2; T)$ independent of m_2

$$\frac{m_1 + m_2}{m_1}[f_s(J_1, J_2; T) - f_s(0, J_2; T)] = \int_0^{\frac{1}{2}\pi} \frac{\mathrm{d}\theta}{m_1 \pi}[\ln \mathcal{I}_1 - \ln(1 + g_2)].$$

Behavior of the coupled system near T_{2c}

Near T_{2c}, $\xi_1(T)$ small. When $m_1/\xi_1(T) >> 1$, drop $\alpha_1^{-m_1}$

$$W = \alpha_1^{m_1}\tfrac{1}{2}\mathcal{I}_2, \qquad U^+(t_1, m_1) = \alpha_1^{m_1}\tfrac{1}{2}(1+g_1),$$
$$\mathcal{I}_2 = [(\alpha_2^{m_2} + \alpha_2^{-m_2}) + (\alpha_2^{m_2} - \alpha_2^{-m_2})G(t_1, t_2; \omega)].$$

$C_2(J_1, J_2; T)$ independent of m_1

$$\frac{m_1+m_2}{m_2}[f_s(J_1, J_2; T) - f_s(J_1, 0; T)] = \int_0^{\frac{1}{2}\pi} \frac{\mathrm{d}\theta}{m_2\pi}[\ln \mathcal{I}_2 - \ln(1+g_1)].$$

Difference

$$G(t_1, t_2; \omega) = \frac{t_1 t_2\sqrt{(1+t_1^2)(1+t_2^2)}}{\sqrt{(t_1^2+\omega^2)(1+t_1^2+\omega^2)(t_2^2+\omega^2)(1+t_2^2+\omega^2)}} + O\Big(\frac{\omega^2}{Y_1 Y_2}\Big)$$
$$\approx t_1\sqrt{(1+t_1^2)}\Big/\sqrt{(t_1^2+\omega^2)(1+t_1^2+\omega^2)} + \cdots, \text{ for } \quad T \sim T_{1c}\,(t_2 > 0),$$
$$\approx -t_2\sqrt{(1+t_2^2)}\Big/\sqrt{(t_2^2+\omega^2)(1+t_2^2+\omega^2)} + \cdots, \text{ for } \quad T \sim T_{2c}\,(t_1 < 0).$$

Enhancement $\mathcal{E}(J_1, J_2; T)$

$$\mathcal{E}(J_1, J_2; T) = C(J_1, J_2; T) - C(J_1, 0; T) - C(0, J_2; \check{T})$$

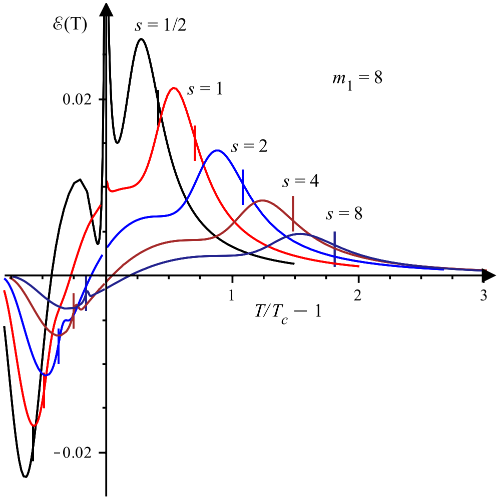
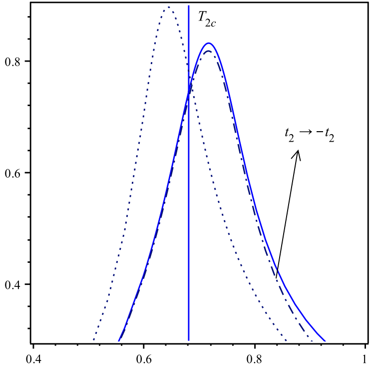


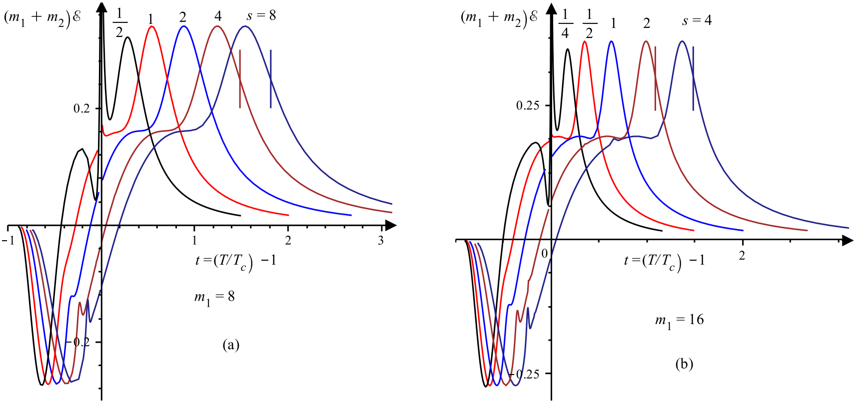
Figure : Plots of $\mathcal{E}(T)$ for $r = 0.3$ and $m_1 = 8$ and various s.



$$\check{T}(T) = T_{2c} - (T - T_{2c}).$$

$$t_2 \to -t_2$$

Enhancement $(m_1 + m_2)\mathcal{E}(J_1, J_2; T)$



The enhancement $\mathcal{E}(t)$: (a) for $m_1 = 8$ and (b) for $m_1 = 16$, but multiplied by $m_1 + m_2$. The short vertical lines locate the corresponding upper limiting critical points, T_{1c}.

$(m_1 + m_2)\mathcal{E}(J_1, J_2; T)$ for fixed $m_2 = 32$ and $r = 0.3$

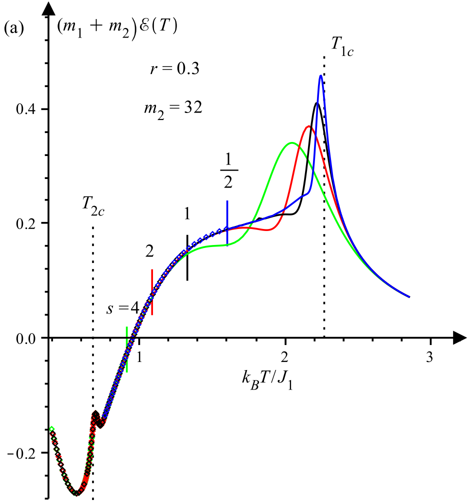


Figure : Plots of the rescaled enhancement for $r = 0.3$, $m_1 = 8, 16, 32, 64$ showing that data collapses occur near T_{2c}.

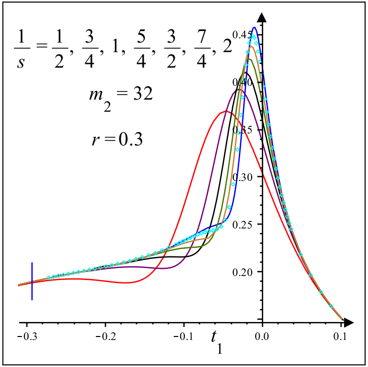


Figure : More detail of behavior near T_{1c} as a function of $t_1 = (T/T_{1c}) - 1$. As m_1 increases, the upper maxima approach T_{1c} from below, and grow steadily in height resembling the corresponding specific heats.

$(m_1 + m_2)\mathcal{E}(J_1, J_2; T)$ for fixed $m_1 = 32$ and $r = 0.3$

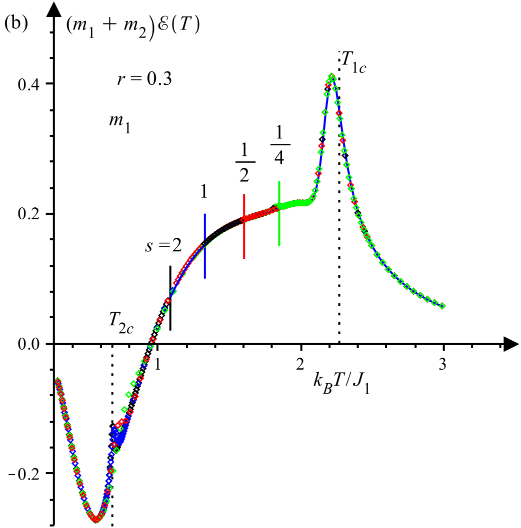


Figure : Plots of the rescaled enhancement for $r = 0.3$, $m_1 = 32$, $m_2 = 8, 16, 32, 64$ showing that data collapses occur near T_{1c}.

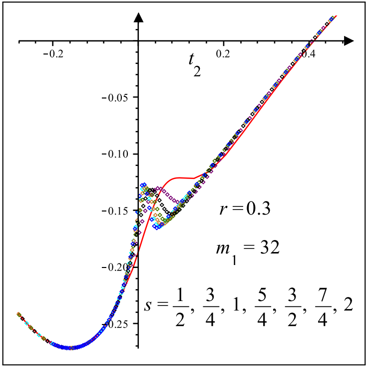


Figure : Rescaled enhancement near T_{2c} plotted as a function of $t_2 = (T/T_{2c}) - 1$. As m_2 increases, the lower maxima approach T_{2c} from above, and grow steadily in height showing logarithmic behavior.

$(m_1 + m_2)\mathcal{E}(J_1, J_2; T)$ for fixed $m_2 = 32$ and $r = 0.5$

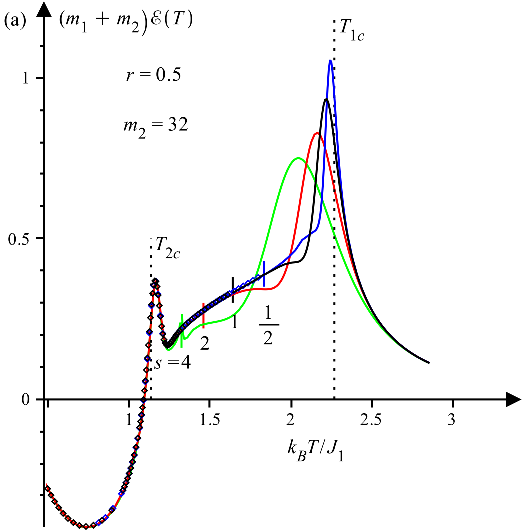


Figure : Plots of the rescaled enhancement for $r = 0.5$ same as $r = 0.3$ for fixed m_2, showing the plots are independent of m_1 near T_{2c}.

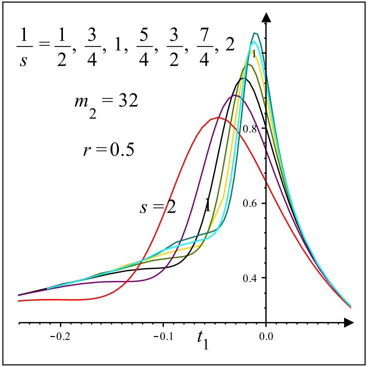


Figure : Detailed behavior of the rescaled enhancement near T_{1c} plotted as a function of $t_1 = (T/T_{1c}) - 1$ which again show the logarithmic behavior.

$(m_1 + m_2)\mathcal{E}(J_1, J_2; T)$ for fixed $m_1 = 32$ and $r = 0.5$

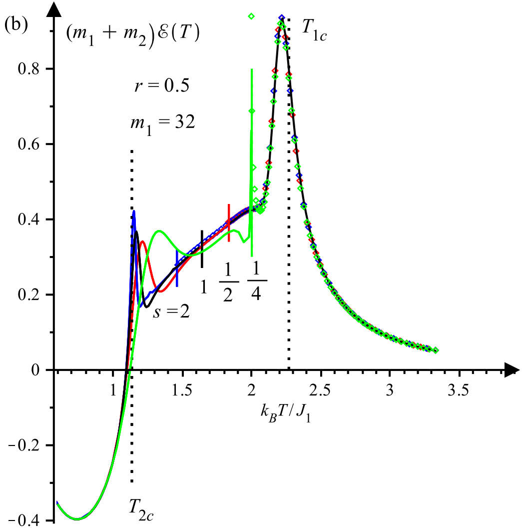


Figure : Plots of the rescaled enhancement for $r = 0.5$, $m_2 = 8, 16, 32, 64$ showing that data collapses occur near T_{1c}.

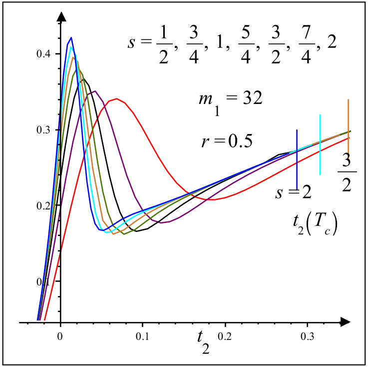


Figure : Behavior near T_{2c} are plotted as functions of $t_2 = (T/T_{2c}) - 1$. The lower maxima again approach T_{2c} from above, and grow steadily in height showing logarithmic behavior.

$(m_1 + m_2)\mathcal{E}(J_1, J_2; T)$ for fixed $m_2 = 32$ and $r = 0.7$

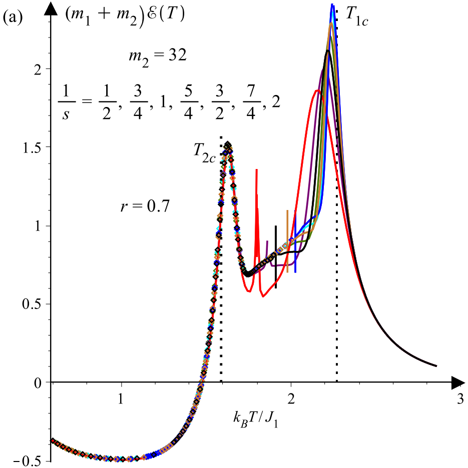


Figure : Plots of the rescaled enhancement for $r = 0.7$, $m_2 = 32$ and $m_1 = 16, 24, \cdots, 64$. Data collapses occur near T_{2c}.

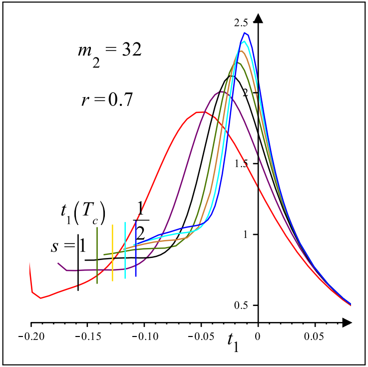


Figure : Detail of behavior of the rescaled enhancement near T_{1c} as a function of $t_1 = (T/T_{1c}) - 1$. As r increases, T_{2c} (denoted by dotted line) and $T_c(r, s)$ (denoted by short vertical lines) move closer to T_{1c},

$(m_1 + m_2)\mathcal{E}(J_1, J_2; T)$ for fixed $m_1 = 32$ and $r = 0.7$

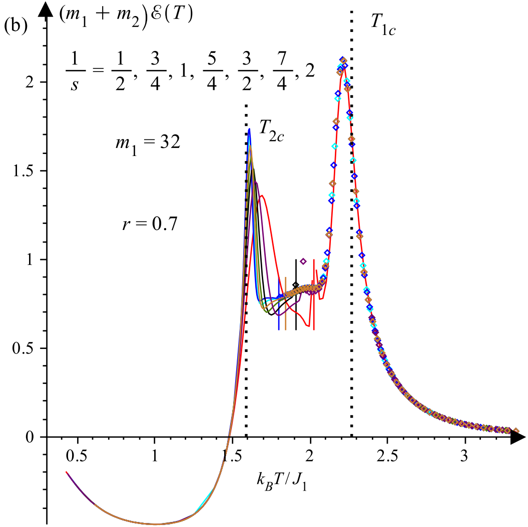


Figure : Plots of the rescaled enhancement for $r = 0.7$, $m_2 = 16, 24, \cdots, 64$ showing that data collapses occur near T_{1c}.

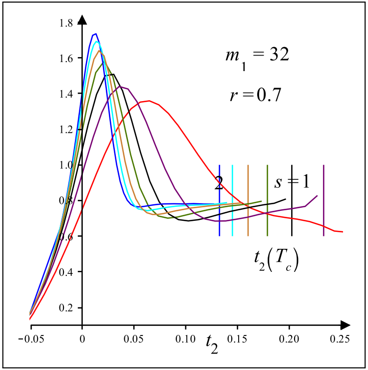


Figure : Detail behavior near T_{2c} as a function of $t_2 = (T/T_{2c}) - 1$, again showing logarithmic divergence.

$(m_1 + m_2)\mathcal{E}(J_1, J_2; T_{ic})$

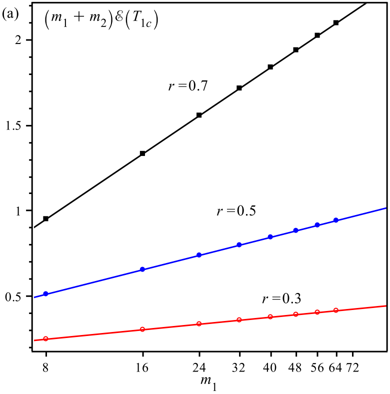


Figure : $(m_1 + m_2)\mathcal{E}(T_{1c}; m_1)$ plotted as function of $\ln m_1$

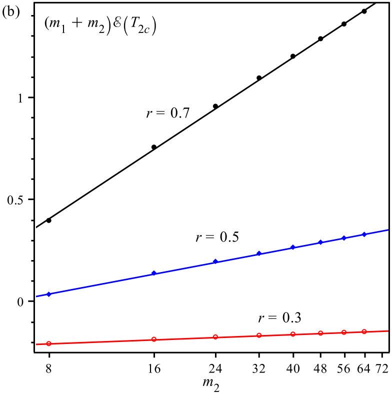


Figure : $(m_1 + m_2)\mathcal{E}(T_{2c}; m_2)$ plotted as function of $\ln m_2$

Summary and Open Questions

Summary

For $J_2 \neq 0$, there is logarithmic divergence at $T_c(r, s)$, but the amplitude decays exponentially.

In agreement with the experiments of Gasparini, the specific heats of the alternating Ising model is enhanced near T_{1c}, the upper limiting critical point; the upper maximum T_{1max} is below T_{1c}, while the lower maximum T_{2max} is above the upper limiting critical point T_{2c} — similar to the the experimental results.

Explicitly, we show under certain conditions that finite-size scaling holds in the vicinity of the upper limiting critical point T_{1c} and and also in the vicinity of the lower critical limit T_{2c}.

Open Question

Can such calculation be done in other models? Can some theoretical conclusion be drawn from these exact calculations for the *proximity effects*?