Eigenvectors for the superintegrable chiral Potts model

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Abstract:

Opening a series of talks on recent work on the superintegrable chiral Potts model, we shall here first briefly review the earlier work, starting with the discovery in 1986 of the first solution of the star-triangle (Yang-Baxter) equation parametrized by a higher genus curve together with a discusssion of the Onsager algebra associated with the superintegrable subcase.

There are two ways to represent the transfer matrix, using either spin variables or bond variables. We shall use the latter approach and construct eigenvectors in a way that resembles the old Ising model work of Onsager and Kaufman, rather than using Bethe Ansatz methods. We shall also outline a strategy to calculate the pair correlation in the general integrable chiral Potts model using only the superintegrable eigenvectors.

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Topics discussed in both talks:

- Chiral Potts model
  - Star-triangle equation
  - Correlation function set up
- Superintegrable case and τ_2 matrix
- Quantum loop algebra and Onsager algebra
- Results for the $Q = 0$ case and for the $Q \neq 0$ case
  - The sl_2 algebra operators
  - The complex rotations
- Combinatorial identities used

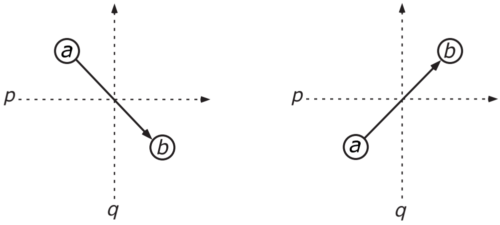
References: H. Au-Yang and J.H.H. Perk, J. Phys. A **41**, 275201 (2008), **42**, 375208 (2009), **43**, 025203 (2009), and arXiv:0907.0362.

Star-Triangle Equation for Spin Models

Onsager—in his 1944 Ising model paper—made a brief remark on *an obvious star-triangle transformation* relating the model on the honeycomb lattice with the one on the triangular lattice.

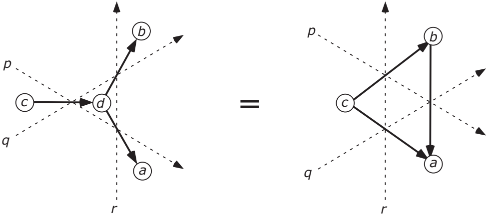
Generalizing, we introduce a lattice with spins $a, b, \cdots = 1, \cdots, N$ on the lattice sites and with interactions between spins a and b given in terms of Boltzmann weight factors $W(a, b)$ and $\overline{W}(a, b)$.

The *integrability* of the model is expressed by the existence of spectral variables (rapidities $p, q, r, \ldots$) that live on oriented lines, drawn dashed here. One can distinguish two kinds of pair interactions depending on the orientations of the spins w.r.t. the *rapidity lines*. Integrability requires that the weights satisfy:

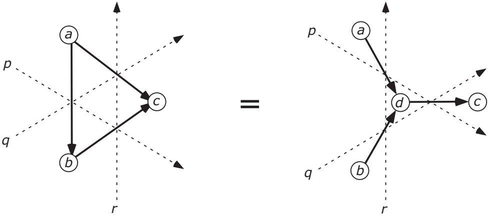


$W_{pq}(a,b)$ $\qquad$ $\overline{W}_{pq}(a,b)$

$$\sum_d \overline{W}_{pq}(c,d)\overline{W}_{qr}(d,b)W_{pr}(d,a) = R_{pqr}\, W_{pq}(b,a)W_{qr}(c,a)\overline{W}_{pr}(c,b)$$



$$R_{pqr}\, W_{pq}(a,b)W_{qr}(a,c)\overline{W}_{pr}(b,c) = \sum_d \overline{W}_{pq}(d,c)\overline{W}_{qr}(b,d)W_{pr}(a,d)$$



These two star-triangle (Yang–Baxter) equations differ only by the negation modulo N of the spin variables a, b, c and d, if the weights satisfy the "Potts condition" $W(a,b) = W(a-b)$ and $\overline{W}(a,b) = \overline{W}(a-b)$. The scalar factor $R(p,q,r)$ can be eliminated by a suitable renormalization of the weights.

Chiral Potts solution of the star-triangle equations

$$W_{pq}(n) = \Big(\frac{\mu_p}{\mu_q}\Big)^n \prod_{j=1}^{n} \frac{y_q - x_p\omega^j}{y_p - x_q\omega^j}$$

$$\overline{W}_{pq}(n) = \big(\mu_p\mu_q\big)^n \prod_{j=1}^{n} \frac{\omega x_p - x_q\omega^j}{y_q - y_p\omega^j}$$

with $\omega \equiv \mathrm{e}^{2\pi\mathrm{i}/N}$, $n = 0, 1, \ldots, N-1$, and (x_p, y_p, μ_p) and (x_q, y_q, μ_q) points of the complex chiral Potts curve (in affine representation)

$$\mu^N = \frac{k'}{1 - kx^N} = \frac{1 - ky^N}{k'}$$

with $0 \le k$, $k' \equiv \sqrt{1-k^2} \le 1$, which follows (after suitable normalization) from the periodicity requirements

$$W(N) = W(0) = 1, \quad \overline{W}(N) = \overline{W}(0) = 1.$$

Curve	Holomorphic differentials	Genus
$\begin{cases} x^N + y^N = k(1+x^N y^N) \\ \mu^N = k'/(1-kx^N) \end{cases}$	$\dfrac{x^{p-1}dx}{(k-x^N)^{q/N}(1-kx^N)^{r/N}}$ $(1 \le p < q+r,\ 0 \le q, r \le N-1)$	$(N-1)(N^2-N-1)$ $(N-1)(N-\tau)(N+\tau^{-1})$ $g=1, 10, 33, 76$ for $N=2,3,4,5$
$x^N + y^N = k(1+x^N y^N)$	$\dfrac{x^{p-1}y^q dx}{k-x^N}$ $(1 \le p, q \le N-1)$	$(N-1)^2$ $g=1, 4, 9, 16$ for $N=2,3,4,5$
$x^N + y^N = 1$ $(x \sim y \sim k^{1/N} \to 0)$	$\dfrac{x^{p-1}dx}{y^q}$ $(1 \le p < q \le N-1)$	$\frac{1}{2}(N-1)(N-2)$ $g=0, 1, 3, 6$ for $N=2,3,4,5$
$x^N + y^N = 0$	None	0

Automorphisms of the chiral Potts curve

Much of the progress has been made using the four special maps

$$\mathrm{R}: \; (x, y, \mu) \;\rightarrow\; \big(y, \omega x, \frac{1}{\mu}\big)$$

$$\mathrm{S}: \; (x, y, \mu) \;\rightarrow\; \big(\frac{1}{y}, \frac{1}{x}, \frac{\omega^{-\frac{1}{2}} y}{x\mu}\big)$$

$$\mathrm{T}: \; (x, y, \mu) \;\rightarrow\; (\omega x, \omega^{-1} y, \omega^{-1}\mu)$$

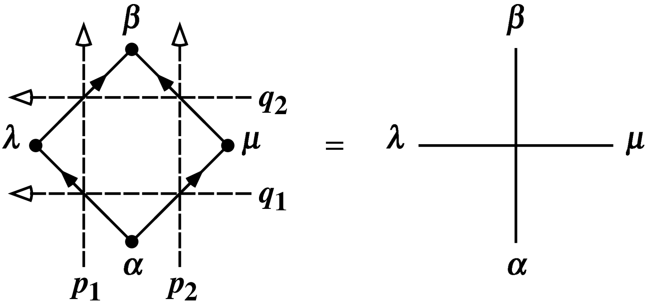
$$\mathrm{U}: \; (x, y, \mu) \;\rightarrow\; (\omega x, y, \mu)$$

which generate the group of $4N^3$ automorphisms that leave the curve

$$\mu^N = \frac{k'}{1 - kx^N} = \frac{1 - ky^N}{k'}$$

invariant.

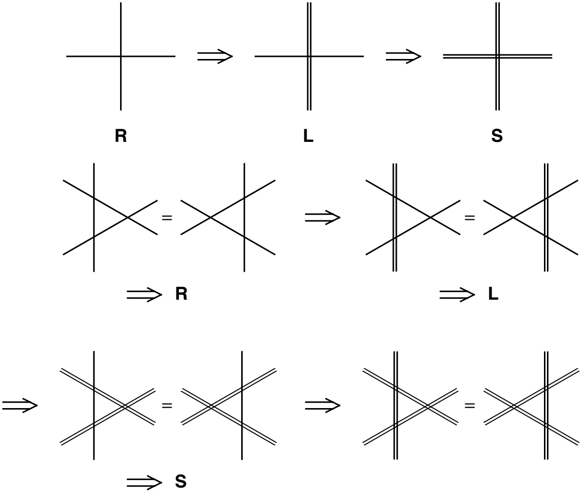
Checkerboard chiral Potts as vertex model



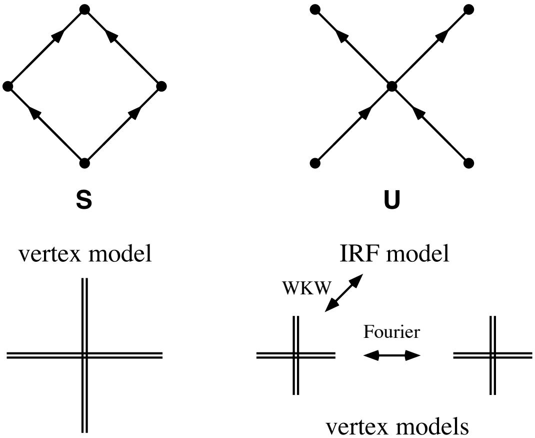
$$R_{\alpha\beta|\lambda\mu} = \overline{W}_{p_1q_1}(\alpha-\lambda)\overline{W}_{p_2q_2}(\mu-\beta)W_{p_2q_1}(\alpha-\mu)W_{p_1q_2}(\lambda-\beta)$$

$$\hat{R}_{\alpha\beta|\lambda\mu} = \frac{s_\beta\, t_\lambda}{s_\alpha\, t_\mu}\,\frac{1}{N^2}\,\sum_{\alpha'=1}^{N}\sum_{\beta'=1}^{N}\sum_{\lambda'=1}^{N}\sum_{\mu'=1}^{N}\omega^{-\alpha\alpha'+\beta\beta'+\lambda\lambda'-\mu\mu'}R_{\alpha'\beta'|\lambda'\mu'}$$

The Bazhanov–Stroganov construction

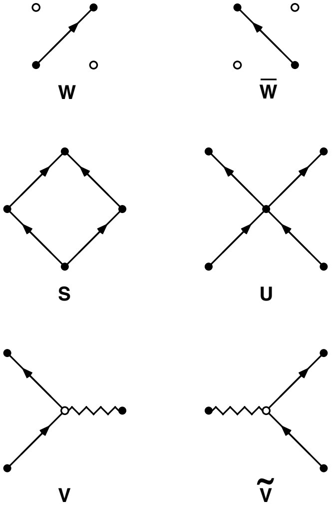


Two direct ways to get to a vertex model



For the "star" on the right we can take the Wu–Kadanoff–Wegner map: Each link state is the difference of the correponding two spin states modulo N.

Elementary units used in the chiral Potts theory



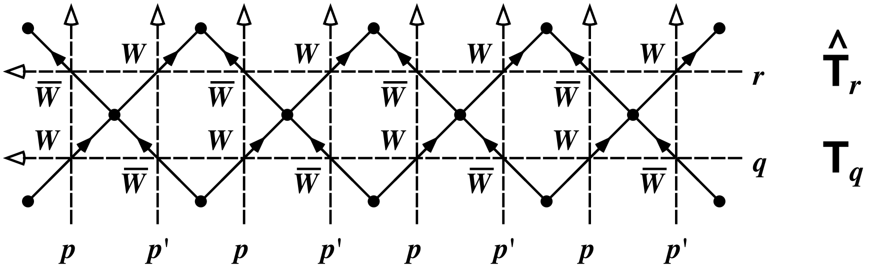
- The original horizontal and vertical pair-interaction Boltzmann weights
- The “square” and the “star” leading to vertex model equivalents. Using these we can make connection (à la Bazhanov–Stroganov) with the more usual set-up of the representatition theory of quantum groups.
- The two “triangles” decomposing the “star.” Here the wiggly lines denote Fourier and inverse Fourier. These triangles were used in our first paper and in the original proof of the star-triangle equations for chiral Potts.

Commuting diagonal transfer matrices

From the star-triangle equations (in the normalization scalar factor $R_{pqr} \equiv 1$) it follows that the diagonal transfer matrices (with periodic boundary conditions) commute with each other and with the Hamiltonian of a quantum spin chain:

$$\mathsf{T}_q \hat{\mathsf{T}}_r = \mathsf{T}_r \hat{\mathsf{T}}_q, \quad [\mathcal{H}, \mathsf{T}_q \hat{\mathsf{T}}_r] = 0.$$

(When $p = p'$ below, with half-shift $\mathsf{S}_{1/2}$ even $[\mathcal{H}, \mathsf{T}_q \mathsf{S}_{1/2}] = [\mathcal{H}, \mathsf{S}_{1/2} \hat{\mathsf{T}}_q] = 0$.)



The functional equations of Baxter–Bazhanov–Perk

Choosing $r = \mathrm{U}^{j+1}\mathrm{R}^{-1}q = q' = (\omega^j y_q, x_q, 1/\mu_q)$, we can derive the splitting

$$\mathsf{T}_q\, \hat{\mathsf{T}}_{q'} = B^{(j)}_{pp'q}\, \mathsf{X}^{-j}\, \tau_j(t_q) + \bar{B}^{(N-j)}_{pp'q}\, \tau_{N-j}(\omega^j t_q), \qquad t_q \equiv x_q y_q.$$

Transfer matrix τ_j is made up of L-operators intertwining a cyclic and a spin $s = \frac{j-1}{2}$ representation, the B's are known scalars, and X is the spin shift operator. Repeating the same process for the τ_j transfer matrices one obtains their fusion relations, for $j = 1, \ldots, N$,

$$\tau_j(t_q)\, \tau_2(\omega^{j-1} t_q) = z(\omega^{j-1} t_q)\, \mathsf{X}\, \tau_{j-1}(t_q) + \tau_{j+1}(t_q),$$

$$\tau_j(\omega t_q)\, \tau_2(t_q) = z(\omega t_q)\, \mathsf{X}\, \tau_{j-1}(\omega^2 t_q) + \tau_{j+1}(t_q),$$

$$\tau_0(t) = 0, \quad \tau_1(t) = \mathbf{1}, \quad \tau_{N+1}(t_q) = z(t_q)\, \mathsf{X}\, \tau_{N-1}(\omega t_q) + (\alpha_q + \bar{\alpha}_q)\, \mathbf{1}.$$

where $z(t)$, α_q and $\bar{\alpha}_q$ are known scalar functions.

Free energy of the chiral Potts model (by Baxter)

$$N f_{pq} = \ln\left[\det_{1\le i,j\le N} \overline{W}(i-j) \prod_{n=0}^{N-1} W_{pq}(n)\right] + \frac{N-1}{2}\ln\frac{\lambda_q}{\lambda_p} + A(\lambda_p, t_q) - A(\lambda_q, t_p) - B(\lambda_p, \lambda_q),$$

$$A(\lambda, t) \equiv \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \frac{1+\lambda \mathrm{e}^{\mathrm{i}\theta}}{1-\lambda \mathrm{e}^{\mathrm{i}\theta}} \sum_{j=1}^{N-1} (N-j)\ln\left[\omega^{-1/2}\Delta(\theta) - \omega^{1/2} t\right],$$

$$B(\lambda, \lambda') \equiv \frac{1}{8\pi^2}\int_0^{2\pi} \mathrm{d}\theta\, \frac{1+\lambda \mathrm{e}^{\mathrm{i}\theta}}{1-\lambda \mathrm{e}^{\mathrm{i}\theta}} \int_0^{2\pi} \mathrm{d}\phi\, \frac{1+\lambda' \mathrm{e}^{\mathrm{i}\phi}}{1-\lambda' \mathrm{e}^{\mathrm{i}\phi}} \times \sum_{j=1}^{N-1} (N-2j)\ln\left[\omega^{-1/2}\Delta(\theta) - \omega^{1/2}\Delta(\phi)\right],$$

$$\Delta(\theta) \equiv \left[\frac{1+k'^2 - 2k'\cos\theta}{k^2}\right]^{1/N}, \quad \lambda_p \equiv \mu_p^N, \quad t_p \equiv x_p y_p.$$

Critical exponents and order parameters of chiral Potts model

In the scaling regime, the free energy behaves as (u_p, u_q are regular functions)

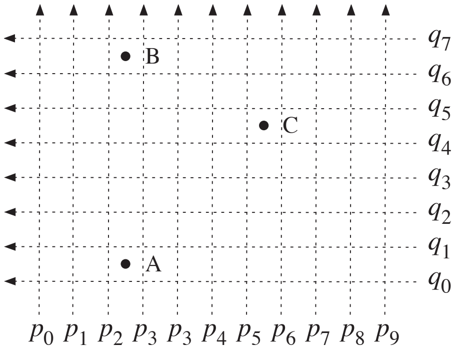
$$f - f_{\mathrm{FZ}} \equiv f - f_{\mathrm{c}} = -\frac{(N-1)k^2}{2N\pi}(u_q - u_p)\cos(u_p + u_q)$$
$$+ \frac{k^2}{4\pi^2}\sin(u_q - u_p)\sum_{j=1}^{\lceil (N-1)/2 \rceil} \frac{\tan(\pi j/N)}{j}\,\mathrm{B}\left(1+\frac{j}{N},\frac{1}{2}\right)^2\left(\frac{k}{2}\right)^{4j/N}$$
$$+\,\mathrm{O}(k^4\log k), \quad \text{if} \quad k^2 \sim T_{\mathrm{c}} - T \to 0, \quad \alpha = 1 - \frac{2}{N}.$$

Albertini, McCoy, Perk, and Tang conjectured that in the ordered state,

$$\langle \omega^{n\sigma_0} \rangle = (1 - {k'}^2)^{\beta_n}, \quad \beta_n = \frac{n(N-n)}{2N^2}, \quad (1 \le n \le N-1, \quad \sigma_0 = 0, \ldots, N-1).$$

This was finally proved by Baxter 17 years later using functional equations.

Baxter's Z-invariance for correlation functions



Pair correlation functions only depend on values of rapidities passing in the same direction between spins $s \equiv \omega^\sigma$ and s'.

This may require us to flip the direction of some rapidities by the automorphism $q \to \mathrm{R}q$, given by $x_q \to y_q$, $y_q \to \omega x_q$.

Find universal functions $g_{2m}^{(n)}$, ($n = 1, \cdots, N-1$, as $s^N = 1$).

$$\langle s_{\mathrm{A}}^{\,n} s_{\mathrm{B}}^{-n} \rangle = g_6^{(n)}(q_1, q_2, q_3, q_4, q_5, q_6)$$

$$\langle s_{\mathrm{A}}^{\,n} s_{\mathrm{C}}^{-n} \rangle = g_8^{(n)}(q_1, q_2, q_3, q_4, p_1, p_2, p_3, p_4)$$

$$\langle s_{\mathrm{C}}^{\,n} s_{\mathrm{B}}^{-n} \rangle = g_6^{(n)}(q_5, q_6, \mathrm{R}p_1, \mathrm{R}p_2, \mathrm{R}p_3, \mathrm{R}p_4)$$

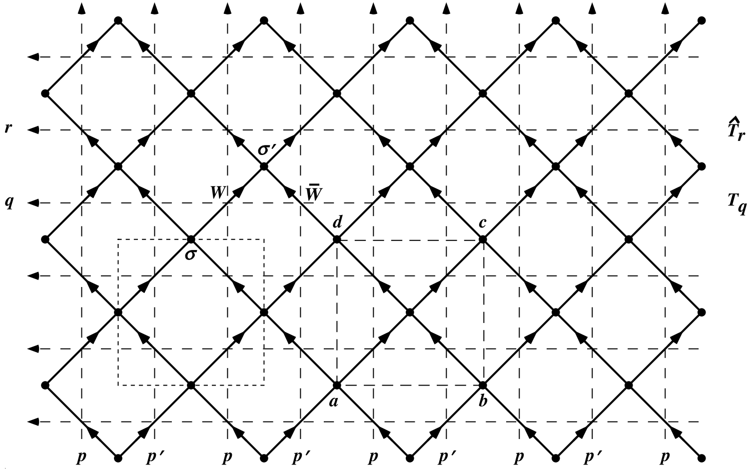
Corollary

The Z-invariance property means that we should be able to calculate correlation functions in the bulk of an infinite general integrable chiral Potts model from special correlations in the much simpler superintegrable case.

For example, a pair correlation function only depends on modulus k and the rapidities passing between the spins. So take an infinite square lattice with special vertical rapidities p and p' alternatingly, but with more general horizontal rapidities $q, r, \cdots$. Choose the two spins within faces in the same vertical column. Such a correlation $\langle s_{10}^{n} s_{1R}^{-n} \rangle$ is independent of p and p'.

General superintegrable chiral Potts model

Assume alternating vertical rapidities $x_{p'} = y_p$, $y_{p'} = x_p$, $\mu_{p'} = 1/\mu_p$:



Quantum chiral Potts spin chain hamiltonian

From the transfer matrix, taking the usual logarithmic derivative, we find

$$\mathcal{H}=\sum_{j=1}^{L}\sum_{n=1}^{N-1}\left[k'\,\frac{\mathrm{e}^{\mathrm{i}(2n-N)\bar\varphi/N}}{\sin(\pi n/N)}\left(\mathrm{X}_j\right)^n+\frac{\mathrm{e}^{\mathrm{i}(2n-N)\varphi/N}}{\sin(\pi n/N)}\left(\mathrm{Z}_j\mathrm{Z}_{j+1}^{\dagger}\right)^n\right],$$

(up to an overall constant factor). The spin operators are

$$\mathrm{X}_j\equiv I_N\otimes I_N\otimes\cdots\otimes\mathrm{X}\otimes\cdots\otimes I_N,\quad \mathrm{Z}_j\equiv I_N\otimes I_N\otimes\cdots\otimes\mathrm{Z}\otimes\cdots\otimes I_N,$$

$$\mathrm{X}\equiv\begin{pmatrix}0&0&0&\dots&0&1\\1&0&0&\dots&0&0\\0&1&0&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&0&0\\0&0&0&\dots&1&0\end{pmatrix},\qquad \mathrm{Z}\equiv\begin{pmatrix}1&0&0&\dots&0&0\\0&\omega&0&\dots&0&0\\0&0&\omega^2&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&\omega^{N-2}&0\\0&0&0&\dots&0&\omega^{N-1}\end{pmatrix}.$$

and $\cos\varphi=k'\cos\bar\varphi$, as $\mathrm{e}^{2\mathrm{i}\varphi/N}=\dfrac{\omega^{1/2}x_p}{y_p}$, $\mathrm{e}^{2\mathrm{i}\bar\varphi/N}=\dfrac{\omega^{1/2}x_p{\mu_p}^2}{y_p}$.

Superintegrability and Onsager algebra

The special choices $p = p'$ lead to greater symmetry.

The 3-state case was noticed by Howes, Kadanoff, and den Nijs. The special N-state case was first derived by von Gehlen and Rittenberg, who wrote the hamiltonian as $\mathcal{H} = A_0 - k' A_1$ and then applied the Dolan–Grady conditions

$$[A_0, [A_0, [A_0, A_1]]] \propto [A_0, A_1], \quad [A_1, [A_1, [A_1, A_0]]] \propto [A_0, A_1].$$

These conditions were shown by Perk (1987) and Davies (1990) to be equivalent to the existence of an Onsager loop algebra

$$\boxed{[A_j, A_k] = 4G_{j-k}, \quad [G_m, A_l] = 2A_{l+m} - 2A_{l-m}, \quad [G_j, G_k] = 0.}$$

This means that we have both star-triangle (Yang–Baxter) integrability and Onsager integrability. Therefore we called this the "superintegrable" case. In the two-dimensional model, the vertical rapidities then are all equal and satisfy $\boxed{x_p^N = y_p^N = \dfrac{1-k'}{k}, \quad \mu_p^{2N} = 1}$. Davies also proved the existence of a minimal n and coefficients $\alpha_k = \pm\alpha_{-k}$, all with same sign, $(k = -n, \ldots, n)$, such that

$$\sum_{k=-n}^{n} \alpha_{\pm k} A_{k-l} = 0, \quad \sum_{k=-n}^{n} \alpha_{\pm k} G_{k-l} = 0, \qquad f(z) \equiv \sum_{k=-n}^{n} \alpha_k z^{k+n}.$$

If all zeros distinct and none equal ± 1,[†] then $z_{-j} = 1/z_j$, $j = 1, \ldots, n$, and

$$\boxed{A_m = 2\textstyle\sum_{j=1}^{n} (z_j^m E_j^+ + z_j^{-m} E_j^-), \quad G_m = 2\textstyle\sum_{j=1}^{n} (z_j^m - z_j^{-m}) H_j}$$

$$\text{with} \quad [E_j^+, E_k^-] = \delta_{jk} H_k, \quad [H_j^+, E_k^{\pm}] = \pm 2\delta_{jk} E_k^{\pm}.$$

To invert this, define

$$f_l^{\pm}(z) \equiv \prod_{\substack{j=1 \\ j\neq l}}^{n} \left(\frac{z - z_{\pm j}}{z_l - z_{\pm j}} \right) \prod_{j=1}^{n} \left(\frac{z - z_{\mp j}}{z_l - z_{\mp j}} \right) = \sum_{k=-n}^{n} \beta_{l\pm,k} z^{k+n}$$

$$\text{with property} \quad f_l^{\pm}(z_{\pm m}) = \delta_{lm}, \quad f_l^{\pm}(z_{\mp m}) = 0, \quad \text{leading to}$$

$$E_l^{\pm} = \frac{1}{2} z_l^{\pm n} \sum_{k=-n}^{n} \beta_{l\pm,k} A_k, \quad H_k = \sum_{l=-n}^{n} \sum_{m=-n}^{n} \beta_{k+,l} \beta_{k-,m} G_{l-m}.$$

[†] Davies also says what happens if this is not true.

Ising-like approach to correlation functions?

What we may hope for without using Bethe Ansatz methods:

- Algebraic derivation of free energy (ground state energy)
- Algebraic derivation of the order parameter(s) [Helen's talks]
- Form factor expansions for correlation functions and susceptibilities
- Integrable recurrence relations for correlation functions
- Painlevé-type differential and difference equations

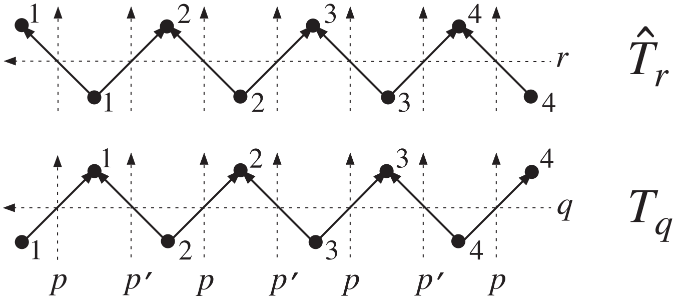
A recent paper using this for the 2d Ising model and 1d transverse Ising chain:

JHHP and H. Au-Yang, J. Stat. Phys. **135** (2009) 599–619 [arXiv:0901.1931].

Another Ising application is the derivation of long high- and low-temperature series for Ising susceptibilities on the triangular, honeycomb and Kagome lattices (joint work in progress with B.G. Nickel, A.J. Guttmann, and Y.-B. Chan).

Correlation functions and order parameters

First we define the transfer matrices (periodic boundary conditions)



$$T_q = T(x_q, y_q)_{\sigma\sigma'} = \prod_{J=1}^{L} W_{pq}(\sigma_J - \sigma'_J)\overline{W}_{p'q}(\sigma_{J+1} - \sigma'_J),$$

$$\hat{T}_r = \hat{T}(x_r, y_r)_{\sigma'\sigma''} = \prod_{J=1}^{L} \overline{W}_{pr}(\sigma'_J - \sigma''_J)W_{p'r}(\sigma'_J - \sigma''_{J+1}).$$

We define the vertical pair correlation functions, between spins in the first column separated by 2ℓ horizontal rapidities, as

$$g_{2\ell}(r;k;q_1,\cdots,q_{2\ell})=\frac{1}{Z}\sum_{\{\sigma\}}\omega^{r(\sigma_{1,0}-\sigma_{1.2\ell})}\prod_{\text{all } W \text{ bonds}}W_{pq}(\sigma,\sigma')\overline{W}_{\bar{p}q}(\sigma,\sigma''),$$

where $r=1,\ldots,N-1$. For Ising we only have $r=1$. We have suppressed the values of the vertical rapidities in our notations. Assuming periodic boundary conditions,

$$g_{2\ell}(r;k;q_1,\cdots,q_{2\ell})=\frac{1}{Z}\operatorname*{Tr}_{\{\sigma\}}\left[\mathrm{Z}_1^r\Big(\prod_{j=1}^{\ell}T_{q_{2j-1}}\hat{T}_{q_{2j}}\Big)\mathrm{Z}_1^{\dagger r}\Big(\prod_{j=\ell+1}^{M}T_{q_{2j-1}}\hat{T}_{q_{2j}}\Big)\right],$$

with $Z=\operatorname*{Tr}_{\{\sigma\}}\Big(\prod_{j=1}^{M}T_{q_{2j-1}}\hat{T}_{q_{2j}}\Big)$. Once the thermodynamic limit $L,M\to\infty$ is taken, this should give the most general pair correlations on infinite Z-invariant lattices.

We defined operators Z and X acting as $\mathrm{Z}|\sigma\rangle = \omega^\sigma|\sigma\rangle$ and $\mathrm{X}|\sigma\rangle = |\sigma+1\rangle$ and L pairs of copies Z_j and X_j, acting in the jth column, $(j = 1,\dots,L)$. The transfer matrices are invariant under the spin shift operator

$$\mathcal{X} \equiv \prod_{j=1}^{L} \mathrm{X}_j, \qquad \mathcal{X}|\sigma_1,\dots,\sigma_L\rangle = |\sigma_1+1,\dots,\sigma_L+1\rangle.$$

The transfer matrix elements only depend on differences $n_j = \sigma_j - \sigma_{j+1}$. So, write

$$|\sigma_1,\dots,\sigma_L\rangle \equiv |\{\sigma_j\}\rangle \equiv |\sigma_1,\{n_j\}\rangle, \quad n_j = \sigma_j - \sigma_{j+1}, \quad \sum_{j=1}^{L} n_j = 0,$$

and define a new basis

$$|Q\,;\{n_j\}\rangle \equiv N^{-1/2} \sum_{\sigma_1=0}^{N-1} \omega^{-Q\sigma_1}|\sigma_1,\{n_j\}\rangle.$$

Then

$$\mathcal{X}\,|\sigma_1,\{n_j\}\rangle = |\sigma_1+1,\{n_j\}\rangle, \quad \text{so that} \quad \boxed{\mathcal{X}\,|Q\,;\{n_j\}\rangle = \omega^Q|Q\,;\{n_j\}\rangle}\,.$$

Using this (Fourier-transformed) basis, we find the transfer matrix elements

$$\langle Q\,;\{n_j'\}|\,T\,|Q\,;\{n_j\}\rangle = \frac{1}{N}\sum_{\sigma_1'=0}^{N-1}\sum_{\sigma_1=0}^{N-1}\omega^{Q(\sigma_1'-\sigma_1)}\langle\,\sigma_1',\{n_j'\}|\,T\,|\sigma_1,\{n_j\}\rangle.$$

As

$$\langle\,\sigma_1',\{n_j'\}|\,T\,|\sigma_1,\{n_j\}\rangle = T(\sigma_1'-\sigma_1,\{n_j'\},\{n_j\}) = T(m,\{n_j'\},\{n_j\}),$$

is a function of differences only, (and $m\equiv\sigma_1'-\sigma_1$), we obtain

$$\langle\,Q\,;\{n_j'\}|\,T\,|Q\,;\{n_j\}\rangle = \sum_{m=0}^{N-1}\omega^{mQ}\,T(m,\{n_j'\},\{n_j\}) \equiv T_Q(\{n_j'\},\{n_j\}).$$

In the spin-shift Q sector the transfer matrices only depend on bond (link) variables. The inverse relation is

$$\langle\{\sigma_j'\}|\,T\,|\{\sigma_j\}\rangle = \frac{1}{N}\sum_{Q=0}^{N-1}\omega^{Q(\sigma_1-\sigma_1')}T_Q(\{n_j'\},\{n_j\}).$$

Matrix products can also be rewritten:

$$\langle\{\sigma_j'\}|\, T\hat{T}\,|\{\sigma_j\}\rangle = \sum_{\sigma_1''=0}^{N-1}\sum_{\{n''\}}\langle\{\sigma_j'\}|\, T\,|\{\sigma_j''\}\rangle\langle\{\sigma_j''\}|\,\hat{T}\,|\{\sigma_j\}\rangle$$

$$= \frac{1}{N^2}\sum_{\sigma_1''=0}^{N-1}\sum_{\{n''\}}\sum_{Q=0}^{N-1}\sum_{Q'=0}^{N-1}\omega^{Q(\sigma_1''-\sigma_1')}T_Q(\{n_j'\},\{n_j''\})\omega^{Q'(\sigma_1-\sigma_1'')}\hat{T}_{Q'}(\{n_j''\},\{n_j\})$$

$$= \frac{1}{N}\sum_{\{n''\}}\sum_{Q=0}^{N-1}\sum_{Q'=0}^{N-1}\omega^{Q(\sigma_1-\sigma_1')}\delta_{Q,Q'}T_Q(\{n_j'\},\{n_j''\})\hat{T}_{Q'}(\{n_j''\},\{n_j\})$$

$$= \frac{1}{N}\sum_{\{n''\}}\sum_{Q=0}^{N-1}\omega^{Q(\sigma_1-\sigma_1')}T_Q(\{n_j'\},\{n_j''\})\hat{T}_Q(\{n_j''\},\{n_j\})$$

$$= \frac{1}{N}\sum_{Q=0}^{N-1}\omega^{Q(\sigma_1-\sigma_1')}\langle\{n_j'\}|\, T_Q\hat{T}_Q\,|\{n_j\}\rangle.$$

For the pair correlation function we find similarly (in the special case of equal horizontal rapidities and using $\langle\{\sigma\}|\mathrm{Z}_1^r|\{\sigma\}\rangle = \omega^{r\sigma_1}$):

$$
\begin{aligned}
&g_{2\ell}(r;k;q,\cdots,q)\\
&\quad= \frac{1}{Z}\operatorname*{Tr}_{\{\sigma\}}\left[\mathrm{Z}_1^r\biggl(\prod_{j=1}^{\ell} T_{q_{2j-1}}\hat{T}_{q_{2j}}\biggr)\mathrm{Z}_1^{\dagger r}\biggl(\prod_{j=\ell+1}^{M} T_{q_{2j-1}}\hat{T}_{q_{2j}}\biggr)\right]\\
&\quad= \frac{1}{Z}\sum_{Q=0}^{N-1}\operatorname*{Tr}_{\{n_j\}}\Bigl\{\Bigl[T_{Q-r}(x_q,y_q)\hat{T}_{Q-r}(x_q,y_q)\Bigr]^{\ell}\Bigl[T_Q(x_q,y_q)\hat{T}_Q(x_q,y_q)\Bigr]^{M-\ell}\Bigr\}
\end{aligned}
$$

Let the eigenvectors of the transfer matrices be given by

$$
T_Q(x_q,y_q)\hat{T}_Q(x_q,y_q)|\mathcal{Y}_j^Q\rangle = \bigl(\Delta_j^Q\bigr)^2|\mathcal{Y}_j^Q\rangle, \qquad \langle\mathcal{Y}_i^Q|\mathcal{Y}_j^Q\rangle = \delta_{i,j},
$$

where Δ_j^Q denotes the jth eigenvalue, and let $\Delta_{\max}^Q$ be the maximum eigenvalue of the transfer matrix T_Q.

In the limit of an infinite number M of rows, the partition function becomes

$$Z = \sum_{Q=0}^{N-1} (\Delta_{\max}^{Q})^2 \to N(\Delta_{\max}^{0})^2, \quad \text{as} \quad L, M \to \infty,$$

as $\Delta_{\max}^{Q}$ for $0 \le Q \le N-1$ are asymptotically degenerate as $L \to \infty$. Therefore,

$$g_{2\ell}(r; k; q, \cdots, q) = \frac{1}{N} \sum_{Q=0}^{N-1} \sum_{j=1}^{J} \left[\frac{\Delta_j^P}{\Delta_{\max}^0} \right]^{2\ell} \langle \mathcal{Y}_{\max}^{Q} | \mathcal{Y}_j^P \rangle \langle \mathcal{Y}_j^P | \mathcal{Y}_{\max}^{Q} \rangle,$$

where $J = N^{L-1}$, $L \to \infty$, and $P \equiv Q - r$.

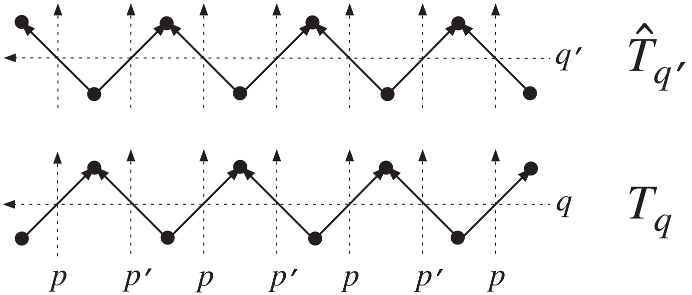
In the limit $\ell \to \infty$, the pair correlation becomes the product of order parameters, i.e.

$$\boxed{\langle \omega^{r\sigma_1} \rangle \langle \omega^{-r\sigma_1} \rangle = \frac{1}{N} \sum_{Q=0}^{N-1} \langle \mathcal{Y}_{\max}^{Q} | \mathcal{Y}_{\max}^{P} \rangle \langle \mathcal{Y}_{\max}^{P} | \mathcal{Y}_{\max}^{Q} \rangle, \quad P \equiv Q - r.}$$

Transfer matrices of "double" superintegrable model

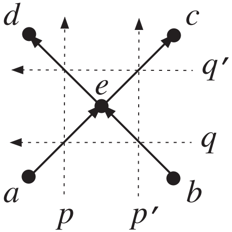
(Superintegrable in both horizontal and vertical directions)

$x_{p'} = y_p,\ y_{p'} = x_p,\ \mu_{p'} = 1/\mu_p$ and $x_{q'} = y_q,\ y_{q'} = \omega^2 x_q,\ \mu_{q'} = 1/\mu_q$:



Their product is block triangular, with diagonal blocks being the superintegrable τ_2 and τ_{N-2} model transfer matrices.

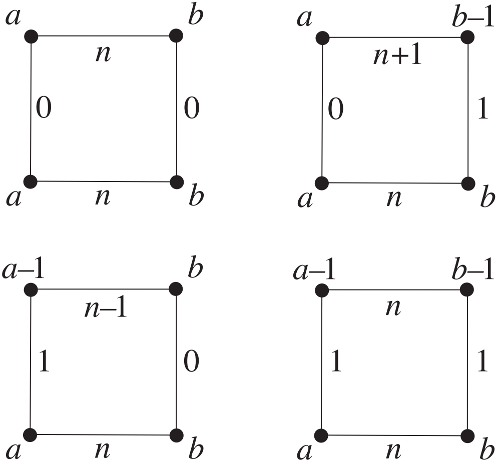
The star weight and the four nonzero τ_2 weights



$U^{(2)}(a,b,c,d)$

$x_{q'} = y_q,\ y_{q'} = \omega^2 x_q$

(p, p' arbitrary if not superintegrable)



Horizontal link variables: $n = a - b \pmod{n}$

Omitting irrelevant factor details, collect the four values in a matrix

$$\mathrm{U}^{(2)} = \begin{pmatrix} 1-\omega^{n+1}t & -\omega t(1-\omega^{n+1}) \\ 1-\omega^{n} & \omega(\omega^{n}-t) \end{pmatrix}, \quad t \equiv \frac{t_q}{t_p} = \frac{x_q y_q}{x_p y_p}, \quad n = a-b$$

We can also keep more of the structure making the elements operators:

$$\mathsf{U}^{(2)} = \begin{pmatrix} \mathbf{1}-\omega t\mathbf{Z} & -\omega t(1-\omega)\mathfrak{f} \\ (1-\omega)\mathfrak{e} & \omega\mathbf{Z}-\omega t\mathbf{1} \end{pmatrix}, \qquad \mathfrak{f} \equiv \frac{\mathbf{1}-\mathbf{Z}}{1-\omega}\,\mathbf{X}, \quad \mathfrak{e} \equiv \mathbf{X}^{-1}\,\frac{\mathbf{1}-\mathbf{Z}}{1-\omega}$$

with operators defined on the basis $\{\,|n\rangle,\ n=0,\ldots,N-1\}$:

$$\mathbf{Z}\,|n\rangle = \omega^{n}\,|n\rangle, \quad \mathbf{X}\,|n\rangle = |n+1\rangle, \qquad |N\rangle \equiv |0\rangle$$

$$\mathfrak{f}\,|n\rangle = [n+1]\,|n+1\rangle, \quad \mathfrak{e}\,|n\rangle = [n]\,|n-1\rangle$$

$$[n] \equiv \frac{1-\omega^{n}}{1-\omega}, \qquad \mathfrak{f}\,|N-1\rangle = \mathfrak{e}\,|0\rangle = 0$$

Monodromy operator and transfer matrix τ_2

Define the operators $\mathbf{Z}_i$, $\mathbf{X}_i$, $\mathfrak{e}_i$, $\mathfrak{f}_i$ and $\mathsf{U}_i^{(2)}$ for link $i = 1, \ldots, L$ by

$$\mathsf{U}_i^{(2)} = \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \underbrace{\mathsf{U}^{(2)}}_{i\mathrm{th}} \otimes \cdots \otimes \mathbf{1}, \quad \text{etc.}$$

Then the monodromy operator is

$$\mathsf{U}(t_q) = \prod_{i=1}^{L} \mathsf{U}_i^{(2)} = \begin{pmatrix} \mathbf{A}(t_q) & \mathbf{B}(t_q) \\ \mathbf{C}(t_q) & \mathbf{D}(t_q) \end{pmatrix} = \sum_{j=0}^{L} (-\omega t)^j \begin{pmatrix} \mathbf{A}_j & \mathbf{B}_j \\ \mathbf{C}_j & \mathbf{D}_j \end{pmatrix}$$

expanding out the polynomial t_q dependence $(t \equiv t_q/t_p)$.

We have many quadratic relations as $\mathsf{U}(t_q)$ satisfies a Yang–Baxter equation with six-vertex model R-matrix intertwiner and

$$\tau_2(t_q) = \operatorname*{tr}_{2\times 2} \mathsf{U}(t_q) = \mathbf{A}(t_q) + \mathbf{D}(t_q)$$

However:

We went from spin variables σ_i to link variables $n_i = \sigma_i - \sigma_{i+1}$, which are invariant under the spin shift $\sigma_i \to \sigma_i + 1$ for all i. Therefore, $\tau_2(t_q)$ has to be block-cyclic. Thus it is better to change basis, i.e. doing spin-Fourier transform, making all matrices block-diagonal, with N blocks, sectors $Q = 0, \ldots, N-1$:

$$\tau_2(t_q)\big|_Q = \mathbf{A}(t_q) + \omega^{-Q}\mathbf{D}(t_q)$$

(using the same letters **A**, **B**, **C**, **D**, but now for the blocks). Coefficients:

$$\mathbf{A}_0 = \mathbf{D}_L = \mathbf{1}, \qquad \mathbf{C}_L = \mathbf{B}_0 = 0, \qquad \mathbf{A}_L = \mathbf{D}_0\,\omega^{-L} = \prod_{j=1}^{L} \mathbf{Z}_j,$$

$$\mathbf{B}_L = (1-\omega)\sum_{j=1}^{L}\Big(\prod_{m=1}^{j-1}\mathbf{Z}_m\Big)\mathfrak{f}_j, \qquad \mathbf{C}_0 = (1-\omega)\sum_{j=1}^{L}\omega^{j-1}\Big(\prod_{m=1}^{j-1}\mathbf{Z}_m\Big)\mathfrak{e}_j,$$

$$\mathbf{B}_1 = (1-\omega)\sum_{j=1}^{L}\omega^{L-j}\mathfrak{f}_j\Big(\prod_{m=j+1}^{L}\mathbf{Z}_m\Big), \quad \mathbf{C}_{L-1} = (1-\omega)\sum_{j=1}^{L}\mathfrak{e}_j\Big(\prod_{m=j+1}^{L}\mathbf{Z}_m\Big).$$

Periodic boundary conditions means $\sigma_{L+1} = \sigma_1 \iff \sum n_i = 0 \bmod N$, which is a restriction on the space of link states $|\{n_i\}\rangle$.

Quantum loop algebra for $Q=0$

Each term in the operators B_1 and B_L raises only one of the n_j's to n_j+1, whereas each term in C_0 and C_{L-1} lowers only one n_j's to n_j-1. To maintain cyclic boundary conditions we must keep $\sum n_j = \ell N$, so we need N of them.

Define

$$\mathbf{B}_1^{(n)} = \frac{(\mathbf{B}_1)^n}{[n]!}, \quad \mathbf{B}_L^{(n)} = \frac{(\mathbf{B}_L)^n}{[n]!},$$

$$\mathbf{C}_0^{(n)} = \frac{(\mathbf{C}_0)^n}{[n]!}, \quad \mathbf{C}_{L-1}^{(n)} = \frac{(\mathbf{C}_{L-1})^n}{[n]!},$$

with $[n]! = [n]\cdots[2]\,[1]$, where $[n] \equiv (1-q^n)/(1-q)|_{q=\omega}$ and $n=1,\ldots,N-1$. This can also be defined this way for $n=N$ using a limit process $q=r\omega,\ r\uparrow 1$.

For $L=\ell N$ we find the loop algebra $\mathcal{L}(\mathfrak{sl}_2)$ generated by

$$\mathbf{x}_0^- = \frac{\mathbf{B}_L^{(N)}}{(1-\omega)^N}, \quad \mathbf{x}_1^- = \frac{\mathbf{B}_1^{(N)}}{(1-\omega)^N},$$

$$\mathbf{x}_0^+ = \frac{\mathbf{C}_0^{(N)}}{(1-\omega)^N}, \quad \mathbf{x}_{-1}^+ = \frac{\mathbf{C}_{L-1}^{(N)}}{(1-\omega)^N}.$$

We then have the $\mathcal{L}(\mathfrak{sl}_2)$ loop algebra

$$\boxed{\mathbf{h}_m = [\mathbf{x}^+_{m-\ell}, \mathbf{x}^-_\ell], \quad \mathbf{x}^\pm_{m+\ell} = \mp\tfrac{1}{2}[\mathbf{h}_m, \mathbf{x}^\pm_\ell], \quad \ell, m \in \mathbb{Z}},$$

after proving the consistency relations (including Serre relations)

$$\begin{aligned}
&\mathbf{h}_0 = [\mathbf{x}^+_0, \mathbf{x}^-_0] = [\mathbf{x}^+_{-1}, \mathbf{x}^-_1],\\
&[\mathbf{h}_0, \mathbf{x}^-_i] = 2\mathbf{x}^-_i, \quad [\mathbf{h}_0, \mathbf{x}^+_{-i}] = -2\mathbf{x}^+_{-i}\\
&[\mathbf{x}^+_{-i}, [\mathbf{x}^+_{-i}, [\mathbf{x}^+_{-i}, \mathbf{x}^-_j]]] = 0, \quad [\mathbf{x}^-_i, [\mathbf{x}^-_i, [\mathbf{x}^-_i, \mathbf{x}^+_{-j}]]] = 0, \quad i \neq j,
\end{aligned}$$

with $i, j = 0, 1$. (Details of the proof can be copied from Deguchi.)

Next define

$$\begin{aligned}
(\mathbf{x}^\pm_m)^{(n)} &\equiv \frac{(\mathbf{x}^\pm_m)^n}{n!}, \quad &&\text{for } n \geq 0,\\
(\mathbf{x}^\pm_m)^{(n)} &\equiv 0, &&\text{for } n < 0,
\end{aligned}$$

with ordinary factorials, not $q = \omega$ factorials. Then:

$$(\mathbf{x}_0^-)^{(n)} = \sum_{\substack{\{0\le\nu_m\le N-1\}\\ \nu_1+\cdots+\nu_L=nN}} \prod_{m=1}^{L} \frac{\mathfrak{f}_j^{\nu_m}}{[\nu_m]!}\, \mathbf{Z}_m^{\sum_{\ell>m}\nu_\ell},$$

$$(\mathbf{x}_0^+)^{(n)} = \sum_{\substack{\{0\le\nu_m\le N-1\}\\ \nu_1+\cdots+\nu_L=nN}} \prod_{m=1}^{L} \mathbf{Z}_m^{\sum_{\ell>m}\nu_\ell}\, \frac{\omega^{m\nu_m}\ \mathfrak{e}_j^{\nu_m}}{[\nu_m]!},$$

$$(\mathbf{x}_1^-)^{(n)} = \sum_{\substack{\{0\le\nu_m\le N-1\}\\ \nu_1+\cdots+\nu_L=nN}} \prod_{m=1}^{L} \frac{\omega^{-m\nu_m}\ \mathfrak{f}_j^{\nu_m}}{[\nu_m]!} \mathbf{Z}_m^{\sum_{\ell<m}\nu_\ell},$$

$$(\mathbf{x}_{-1}^+)^{(n)} = \sum_{\substack{\{0\le\nu_m\le N-1\}\\ \nu_1+\cdots+\nu_L=nN}} \prod_{m=1}^{L} \mathbf{Z}_m^{\sum_{\ell<m}\nu_\ell}\, \frac{\mathfrak{e}_j^{\nu_m}}{[\nu_m]!},$$

where the summations are over the L variables ν_m, for $m = 1, \cdots, L$.

(These identities have been used over and over again, reducing much of the calculation to new summation identities.)

The Drinfeld polynomial

We have three doubly-infinite sequences of finite-dimensional operators $\mathbf{h}_n$, $\mathbf{x}_n^+$ and $\mathbf{x}_n^-$ satisfying

$$\boxed{\mathbf{h}_m = [\mathbf{x}^+_{m-\ell}, \mathbf{x}^-_\ell], \quad \mathbf{x}^\pm_{m+\ell} = \mp\tfrac{1}{2}[\mathbf{h}_m, \mathbf{x}^\pm_\ell], \quad \ell, m \in \mathbb{Z}}.$$

Therefore, there must exist linear relations

$$\sum_{j=0}^{r} \Lambda_j \mathbf{h}_{j+n} = 0, \quad \sum_{j=0}^{r} \Lambda_j \mathbf{x}^+_{j+n} = 0, \quad \sum_{j=0}^{r} \Lambda_j \mathbf{x}^-_{j+n} = 0.$$

We collect the coefficients in the "Drinfeld polynomial"

$$P(z) = \sum_{j=0}^{r} \Lambda_j z^j = \prod_{j=1}^{r} (z - z_j), \quad \text{with } \Lambda_{r-j} = \Lambda_j \text{ for } Q = 0.$$

From Baxter's work we can conclude, also for $Q \neq 0$,

$$P(t^N) = \frac{t^{-Q}}{N} \sum_{n=0}^{N-1} \omega^{-nQ} \frac{(1 - t^N)^L}{(1 - \omega^n t)^L}.$$

This can also be found as follows: The coefficients for $Q=0$ are given by

$$\Lambda_n = \sum_{\substack{\{0\le\nu_m\le N-1\} \\ \nu_1+\cdots+\nu_L=nN}} 1 \equiv \lambda_{nN}, \qquad \lambda_j \equiv \sum_{\substack{\{0\le\nu_m\le N-1\} \\ \nu_1+\cdots+\nu_L=j}} 1,$$

and can be extracted from the generating function

$$\mathcal{Q}(t) = \prod_{m=1}^{L} \left(\sum_{\nu_m=0}^{N-1} t^{\nu_m} \right) = \frac{(1-t^N)^L}{(1-t)^L},$$

where we have inserted t^{ν_m} in each of the L sums in λ_j to arrive at $\mathcal{Q}(t)$. The condition $\nu_1+\cdots+\nu_L=nN$ means Λ_n is the coefficient of t^{nN} in the expansion of $\mathcal{Q}(t)$. This way we find explicitly

$$\Lambda_n = \Lambda_{r-n} = \sum_{m=0}^{n} (-1)^m \binom{L}{m} \frac{(L)_{nN-mN}}{(nN-mN)!}, \quad \text{for } Q=0.$$

Picking every Nth coefficient of $\mathcal{Q}(t)$, beginning with t^Q, we find the Drinfeld polynomials for $Q=0,\ldots,N-1$, of the previous page.

The factorization into $\mathfrak{sl}_2$ algebras

Introduce the polynomials

$$f_j(z) = \prod_{\ell \neq j} \frac{z - z_\ell}{z_j - z_\ell} = \sum_{n=0}^{r-1} \beta_{j,n} z^n, \quad f_j(z_k) = \delta_{j,k}\,,$$

where the $\beta_{j,n}$ are elements of the inverse of a Vandermonde matrix,

$$\sum_{n=0}^{r-1} \beta_{j,n} z_k^n = \delta_{j,k}, \qquad \sum_{k=1}^{r} z_k^n \beta_{k,m} = \delta_{n,m} \quad \text{for} \quad 0 \leq n \leq r-1.$$

Then, following Davies, we can solve the linear dependence as

$$\mathbf{x}_n^{\mp} = \pm \sum_{m=1}^{r} z_m^{-n}\, \mathbf{E}_m^{\pm}, \quad \mathbf{h}_n = \sum_{m=1}^{r} z_m^{-n}\, \mathbf{H}_m,$$

and invert it as

$$\boxed{\mathbf{E}_m^{\pm} = \pm \sum_{n=0}^{r-1} \beta_{m^*,n} z_m^{\ell}\, \mathbf{x}_{n+\ell}^{\mp}, \quad \mathbf{H}_m = \sum_{n=0}^{r-1} \beta_{m^*,n} z_m^{\ell}\, \mathbf{h}_{n+\ell}, \quad m = 1, \ldots, r},$$

where m^* is the index for which $z_{m^*} = 1/z_m$. It is easy to show

$$[\mathbf{E}_m^+, \mathbf{E}_n^-] = \delta_{m,n}\mathbf{H}_m, \quad [\mathbf{H}_m, \mathbf{E}_n^\pm] = \pm 2\delta_{m,n}\mathbf{E}_m^\pm,$$

and all other commutators zero: We have r independent $\mathfrak{sl}(2)$'s.

We have to continue from here tomorrow. We have found a factorization of the algebra for the $Q = 0$ ground state sector. For the Ising case this corresponds to a major part of Onsager's 1944 paper. Several things remain to be done: To obtain the corresponding eigenvectors of the transfer matrix, we need to find rotations à la Onsager. We will also derive results for the $Q \neq 0$ sector, together with combinatorial identities.

End of part 1