Eigenvectors for the superintegrable chiral Potts model II

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Abstract:

Opening a series of talks on recent work on the superintegrable chiral Potts model, we shall here first briefly review the earlier work, starting with the discovery in 1986 of the first solution of the star-triangle (Yang-Baxter) equation parametrized by a higher genus curve together with a discusssion of the Onsager algebra associated with the superintegrable subcase.

There are two ways to represent the transfer matrix, using either spin variables or bond variables. We shall use the latter approach and construct eigenvectors in a way that resembles the old Ising model work of Onsager and Kaufman, rather than using Bethe Ansatz methods. We shall also outline a strategy to calculate the pair correlation in the general integrable chiral Potts model using only the superintegrable eigenvectors.

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Topics discussed in both talks:

- Chiral Potts model
    - ∗ Star-triangle equation
    - ∗ Correlation function set up
- Superintegrable case and τ_2 matrix
- Quantum loop algebra and Onsager algebra
- Results for the $Q = 0$ case and for the $Q \neq 0$ case
    - ∗ The sl_2 algebra operators
    - ∗ The complex rotations
- Combinatorial identities used

References: H. Au-Yang and J.H.H. Perk, J. Phys. A **41**, 275201 (2008), **42**, 375208 (2009), **43**, 025203 (2009), and arXiv:0907.0362.

In our first lecture we introduced the $\mathcal{L}(\mathfrak{sl}_2)$ loop algebra

$$\mathbf{h}_m = [\mathbf{x}^+_{m-\ell}, \mathbf{x}^-_\ell], \quad \mathbf{x}^\pm_{m+\ell} = \mp\tfrac{1}{2}[\mathbf{h}_m, \mathbf{x}^\pm_\ell], \quad \ell, m \in \mathbb{Z},$$

to describe the $Q = 0$ ground state sector of the superintegrable chiral Potts model. We also defined

$$(\mathbf{x}^\pm_m)^{(n)} \equiv \frac{(\mathbf{x}^\pm_m)^n}{n!}, \quad \text{for } n \geq 0,$$

$$(\mathbf{x}^\pm_m)^{(n)} \equiv 0, \quad \text{for } n < 0.$$

These were given in terms of the following operators on horizontal bond states:

$$\mathbf{Z}\,|n\rangle = \omega^n\,|n\rangle, \quad \mathbf{X}\,|n\rangle = |n+1\rangle, \quad |N\rangle \equiv |0\rangle,$$

$$\mathfrak{f}\,|n\rangle = [n+1]\,|n+1\rangle, \quad \mathfrak{e}\,|n\rangle = [n]\,|n-1\rangle,$$

$$[n] \equiv \frac{1 - \omega^n}{1 - \omega}, \quad \mathfrak{f}\,|N-1\rangle = \mathfrak{e}\,|0\rangle = 0,$$

where $n = 0, \dots, N-1$. More precisely, we have L copies of these, $\mathbf{Z}_m$, $\mathfrak{f}_m$, $\mathfrak{e}_m$, $(m = 1, \dots, L)$.

In terms of these we gave the very explicit fundamental formulae:

$$(\mathbf{x}_0^-)^{(n)} = \sum_{\substack{\{0 \le \nu_m \le N-1\} \\ \nu_1 + \cdots + \nu_L = nN}} \prod_{m=1}^{L} \frac{\mathbf{f}_j^{\nu_m}}{[\nu_m]!} \, \mathbf{z}_m^{\sum_{\ell > m} \nu_\ell},$$

$$(\mathbf{x}_0^+)^{(n)} = \sum_{\substack{\{0 \le \nu_m \le N-1\} \\ \nu_1 + \cdots + \nu_L = nN}} \prod_{m=1}^{L} \mathbf{z}_m^{\sum_{\ell > m} \nu_\ell} \, \frac{\omega^{m \nu_m} \, \mathbf{e}_j^{\nu_m}}{[\nu_m]!},$$

$$(\mathbf{x}_1^-)^{(n)} = \sum_{\substack{\{0 \le \nu_m \le N-1\} \\ \nu_1 + \cdots + \nu_L = nN}} \prod_{m=1}^{L} \frac{\omega^{-m \nu_m} \, \mathbf{f}_j^{\nu_m}}{[\nu_m]!} \mathbf{z}_m^{\sum_{\ell < m} \nu_\ell},$$

$$(\mathbf{x}_{-1}^+)^{(n)} = \sum_{\substack{\{0 \le \nu_m \le N-1\} \\ \nu_1 + \cdots + \nu_L = nN}} \prod_{m=1}^{L} \mathbf{z}_m^{\sum_{\ell < m} \nu_\ell} \, \frac{\mathbf{e}_j^{\nu_m}}{[\nu_m]!},$$

where the summations are over the L variables ν_m, for $m = 1, \cdots, L$.

Finally we introduced the Drinfeld polynomial. For $Q = 0$ we found

$$P(t^N) = \frac{1}{N} \sum_{n=0}^{N-1} \frac{(1-t^N)^L}{(1-\omega^n t)^L} = \prod_{j=1}^{r} (z - z_j),$$

for the ground state sector and $r = (N-1)L/N$ if L a multiple of N. Using the zeroes, we also formed the polynomials

$$f_j(z) = \prod_{\ell \neq j} \frac{z - z_\ell}{z_j - z_\ell} = \sum_{n=0}^{r-1} \beta_{j,n} z^n, \quad f_j(z_k) = \delta_{j,k}.$$

This gave the following decomposition à la Onsager and Davies

$$\mathbf{x}_n^\mp = \pm \sum_{m=1}^{r} z_m^{-n} \mathbf{E}_m^\pm, \quad \mathbf{h}_n = \sum_{m=1}^{r} z_m^{-n} \mathbf{H}_m,$$

with the inverse relations

$$\mathbf{E}_m^\pm = \pm \sum_{n=0}^{r-1} \beta_{m^*,n} z_m^\ell \mathbf{x}_{n+\ell}^\mp, \quad \mathbf{H}_m = \sum_{n=0}^{r-1} \beta_{m^*,n} z_m^\ell \mathbf{h}_{n+\ell}, \quad m = 1, \ldots, r,$$

where m^* is the index for which $z_{m^*} = 1/z_m$. It is easy to show

$$[\mathbf{E}_m^+, \mathbf{E}_n^-] = \delta_{m,n} \mathbf{H}_m, \quad [\mathbf{H}_m, \mathbf{E}_n^\pm] = \pm 2\delta_{m,n} \mathbf{E}_m^\pm,$$

and all other commutators zero: We have r independent $\mathfrak{sl}(2)$'s.

We introduce the "ferromagnetic state" $|\Omega\rangle \equiv |\{n_j=0\}\rangle$ with all spins equal and the "antiferromagnetic state" $|\bar\Omega\rangle \equiv |\{n_j=N-1\}\rangle$. We can show

$$\mathbf{H}_m|\Omega\rangle = -|\Omega\rangle, \quad (\mathbf{E}_m^+)^2|\Omega\rangle = 0, \quad \text{for all } m = 1, \ldots, r.$$

Following Onsager we can then define the 2^r states

$$\Psi(\xi_1, \xi_2, \cdots, \xi_r) = \prod_{m \in J_n} \mathbf{E}_m^+|\Omega\rangle, \quad \xi_j = \pm 1,$$

where J_n is any subset of $\{1, 2, \cdots, r\}$, $|J_n| = n$, and

$$\begin{cases} \xi_j = 1, & \text{for } j \in J_n, \\ \xi_j = -1, & \text{for } j \notin J_n, \end{cases}$$

and see tht they form a basis for the ground state sector of the superintegrable τ_2 matrix and the superintegrable chiral Potts model.

From the commutation relations,

$$[\mathbf{E}_m^+, \mathbf{E}_n^-] = \delta_{m,n}\mathbf{H}_m, \quad [\mathbf{H}_m, \mathbf{E}_n^\pm] = \pm 2\delta_{m,n}\mathbf{E}_m^\pm, \quad [\mathbf{E}_m^+, \mathbf{E}_n^+] = [\mathbf{E}_m^-, \mathbf{E}_n^-] = 0,$$

we find

$$\mathbf{H}_m\mathbf{E}_n^+|\Omega\rangle = -(-1)^{\delta_{m,n}}\mathbf{E}_n^+|\Omega\rangle.$$

The ground state sector is then given by the basis

$$\Psi(\xi_1, \xi_2, \cdots, \xi_r) = \prod_{m \in J_n} \mathbf{E}_m^+|\Omega\rangle, \quad \xi_j = \pm 1,$$

where $J_n = (j_1, \cdots, j_n)$ is any subset of $(1, 2, \cdots, r)$ for $n = 0, \cdots, r$, with $\xi_j = 1$ for $j \in J_n$, and $\xi_j = -1$ for $j \notin J_n$. Equivalently,

$$\mathbf{H}_j\Psi(\xi_1, \xi_2, \cdots, \xi_r) = \xi_j\Psi(\xi_1, \xi_2, \cdots, \xi_r).$$

Particularly, for $n = 0$ and $n = r$, we have

$$|\Omega\rangle = \Psi(-1, -1, \cdots, -1), \qquad |\bar{\Omega}\rangle = \Psi(1, 1, \cdots, 1).$$

Because $\xi_j = \pm 1$, it is easy to see that $\mathbf{H}_j^2 = \mathbf{1}$ in this restricted eigenspace. Since $\mathbf{E}_j^- |\Omega\rangle = 0$, and $\mathbf{H}_j |\Omega\rangle = -|\Omega\rangle$, we find
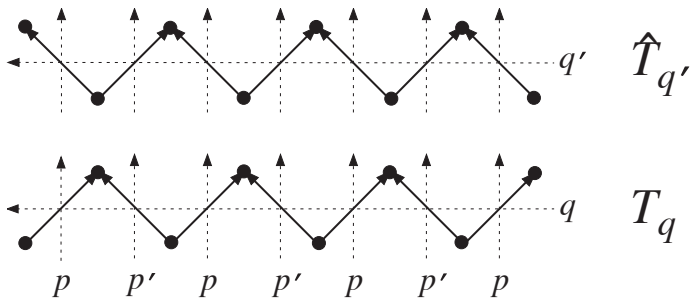
$$\mathbf{E}_j^- \Psi(\xi_1, \xi_2, \cdots, \xi_r) = \mathbf{E}_j^- \prod_{m \in J_n} \mathbf{E}_m^+ |\Omega\rangle$$

$$= \begin{cases} \Psi(\xi_1, \cdots, \xi_{j-1}, -\xi_j, \xi_{j+1}, \cdots, \xi_r) & \text{if } j \in J_n, \\ 0 & \text{if } j \notin J_n. \end{cases}$$

Likewise, using $(\mathbf{E}_j^+)^2 |\Omega\rangle = 0$ we obtain

$$\mathbf{E}_j^+ \Psi(\xi_1, \xi_2, \cdots, \xi_r) = \mathbf{E}_j^+ \prod_{m \in J_n} \mathbf{E}_m^+ |\Omega\rangle$$

$$= \begin{cases} 0 & \text{if } j \in J_n, \\ \Psi(\xi_1, \cdots, \xi_{j-1}, -\xi_j, \xi_{j+1}, \cdots, \xi_r) & \text{if } j \notin J_n. \end{cases}$$

We thus get the usual spin-$\frac{1}{2}$ relations.

Eigenvectors of the superintegrable chiral Potts transfer matrix

Following Baxter, we write

$$T_q = N^{\frac{1}{2}L} \frac{(x_q - y_p)^L}{(x_q^N - y_p^N)^L} \, \mathcal{T}(x_q, y_q), \quad \hat{T}_q = N^{\frac{1}{2}L} \frac{(x_q - x_p)^L}{(x_q^N - x_p^N)^L} \, \hat{\mathcal{T}}(x_q, y_q).$$

taking out "trivial factors."

We need to solve the eigenvalue problem

$$\mathcal{T}_0(x_q, y_q) \,|\mathbf{x}\rangle = \mathcal{G}(\lambda_q) \,|\mathbf{y}\rangle, \quad \hat{\mathcal{T}}_0(x_q, y_q) \,|\mathbf{y}\rangle = \mathcal{G}(\lambda_q) \,|\mathbf{x}\rangle.$$

From Baxter's work (or the spin-$\frac{1}{2}$ structure) we know that for the sector studied

$$\boxed{\mathcal{G}(\lambda_q) = \prod_{j=1}^{r} \left(A_j \pm B_j \right)},$$

where

$$A_j = \rho \cosh \theta_j (1 - \lambda_q^{-1}), \quad B_j = \rho \sinh \theta_j (1 + \lambda_q^{-1}),$$

$$2 \cosh 2\theta_j = k' + k'^{-1} - k^2 t_p^N z_j / k', \quad \rho^r = N^{\frac{1}{2}} (k'/k^2)^{\frac{1}{2} r}.$$

Thus we need rotations $\mathcal{R}$ and $\mathcal{S}$, such that $|\mathbf{x}\rangle = \mathcal{R} \,|\Omega\rangle$, $|\mathbf{y}\rangle = \mathcal{S} \,|\Omega\rangle$, and

$$\boxed{\mathcal{T}_0(x_q, y_q) = \mathcal{S} \prod_{j=1}^{r} \left(A_j - \mathbf{H}_j B_j \right) \mathcal{R}^{-1}, \quad \hat{\mathcal{T}}_0(x_q, y_q) = \mathcal{R} \prod_{j=1}^{r} \left(A_j - \mathbf{H}_j B_j \right) \mathcal{S}^{-1}},$$

or, before the rotations,

$$\mathcal{T}_0(x_q, y_q) = \prod_{j=1}^{r} \left[X_j - \mathbf{H}_j Y_j + (\mathbf{E}_j^+ + \mathbf{E}_j^-) Z_j \right],$$

$$\hat{\mathcal{T}}_0(x_q, y_q) = \prod_{j=1}^{r} \left[X_j' - \mathbf{H}_j Y_j' + (\mathbf{E}_j^+ + \mathbf{E}_j^-) Z_j' \right].$$

After some tedious calculations we found

$$\frac{\langle \Omega | \, \mathcal{T}_0(x_q, y_q) \, | \Omega \rangle}{\langle \Omega | \, \mathbf{E}_m^- \mathcal{T}_0(x_q, y_q) \, | \Omega \rangle} = \frac{X_m + Y_m}{Z_m} = \frac{x_q^N - y_p^N z_m}{x_q^N - y_p^N},$$

$$\frac{\langle \bar{\Omega} | \, \mathcal{T}_0(x_q, y_q) \, | \bar{\Omega} \rangle}{\langle \bar{\Omega} | \, \mathbf{E}_m^+ \mathcal{T}_0(x_q, y_q) \, | \bar{\Omega} \rangle} = \frac{X_m - Y_m}{Z_m} = \frac{x_p^N - y_q^N z_m^{-1}}{x_p^N - y_q^N},$$

and similarly for $\hat{\mathcal{T}}_0(x_q, y_q)$. We find a solution of the form

$$\mathcal{R} = \prod_{j=1}^{r} \mathcal{R}_j, \quad \mathcal{S} = \prod_{j=1}^{r} \mathcal{S}_j, \quad \mathcal{R}_j = (\mathcal{S}_j^{-1})^{\mathrm{t}},$$

with

$$\mathcal{S}_j = \tfrac{1}{2}(s_{11} + s_{22})\mathbf{1} + \tfrac{1}{2}(s_{11} - s_{22})\mathbf{H}_j + s_{12}\mathbf{E}_j^+ + s_{21}\mathbf{E}_j^-\,,$$

with

$$s_{22} = \sqrt{\frac{m_{22}\mathrm{e}^{\theta_j} + n_{22}\mathrm{e}^{-\theta_j}}{2\sinh 2\theta_j}}, \quad s_{12} = \frac{m_{12}\mathrm{e}^{\theta_j} + n_{12}\mathrm{e}^{-\theta_j}}{m_{22}\mathrm{e}^{\theta_j} + n_{22}\mathrm{e}^{-\theta_j}}\, s_{22},$$

$$s_{21} = \frac{\mathrm{e}^{-2\theta_j} - k'}{2s_{12}\sinh 2\theta_j}, \qquad s_{11} = \frac{\mathrm{e}^{2\theta_j} - k'}{2s_{22}\sinh 2\theta_j},$$

and

$$m_{11} = -\bar{\epsilon}_j k' \lambda_p / z_j, \qquad m_{12} = m_{21} = -\bar{\epsilon}_j k' \lambda_p, \qquad m_{22} = \bar{\epsilon}_j(z_j - 1 - k' z_j \lambda_p),$$

$$n_{11} = \bar{\epsilon}_j(z_j^{-1}\lambda_p - \lambda_p + k'), \qquad n_{12} = n_{21} = n_{22} = \bar{\epsilon}_j k',$$

$$\bar{\epsilon}_j^2 \equiv \frac{1}{k'(z_j^{-1} - 1)\lambda_p}.$$

To give some details: We rewrite the transfer matrices in terms of the edge variables $\{n_i\}$ and $\{n'_i\}$ with $n_i = \sigma_i - \sigma_{i+1}$ as

$$\langle\{n'_i\}|\mathcal{T}_Q(x_q, y_q)|\{n_i\}\rangle = N^{-\frac{1}{2}L} \sum_{a=0}^{N-1} \omega^{-Qa} \prod_{j=1}^{L} \left[(y_p^N - x_q^N)(y_p - x_q)^{-1} \right.$$

$$\left. \times W_{pq}(a - N_j + N'_j)\overline{W}_{p'q}(a - N_{j+1} + N'_j) \right], \qquad N_j = \sum_{\ell < j} n_\ell = \sigma_1 - \sigma_j,$$

where $a = \sigma_1 - \sigma'_1$, and

$$W_{pq}(n) = \left(\frac{\mu_p}{\mu_q}\right)^n \prod_{j=1}^{n} \frac{y_q - x_p \omega^j}{y_p - x_q \omega^j}, \quad \overline{W}_{p'q}(n) = \left(\frac{\mu_q}{\mu_p}\right)^n \prod_{j=1}^{n} \frac{\omega y_p - x_q \omega^j}{y_q - x_p \omega^j}.$$

Cancelling out the common factors in the weights,

$$\langle\{n_i'\}|\mathcal{T}_Q(x_q, y_q)|\{n_i\}\rangle = N^{-\frac{1}{2}L}\sum_{a=0}^{N-1}\omega^{-Qa}$$

$$\times \prod_{j=1}^{L}\left[\frac{(y_p^N - x_q^N)\omega^{a-N_{j+1}+N_j'}}{y_p - x_q\omega^{a-N_{j+1}+N_j'}}\left(\frac{\mu_p}{\mu_q}\right)^{n_j}\prod_{\ell=1}^{n_j}\frac{y_q - x_p\omega^{\ell+a-N_{j+1}+N_j'}}{y_p - x_q\omega^{\ell+a-N_{j+1}+N_j'}}\right].$$

It is easy to see that for the two ground states,

$$|\Omega\rangle \leftrightarrow n_i \equiv N_i \equiv 0 \quad \text{or} \quad |\bar{\Omega}\rangle \leftrightarrow n_i \equiv N-1, \ N_i \equiv (i-1)(N-1),$$

the above expression simplifies to

$$\langle\{n_i\}|\mathcal{T}_Q(x_q, y_q)|\Omega\rangle = N^{-\frac{1}{2}L}\sum_{a=0}^{N-1}\omega^{-Qa}\prod_{j=1}^{L}\frac{\omega^{a+N_j}(y_p^N - x_q^N)}{y_p - x_q\omega^{a+N_j}},$$

and

$$\langle\{n_i\}|\mathcal{T}_Q(x_q, y_q)|\bar{\Omega}\rangle = N^{-\frac{1}{2}L}\sum_{a=0}^{N-1}\omega^{-Qa}\prod_{j=1}^{L}\left(\frac{\mu_p}{\mu_q}\right)^{N-1}\frac{\omega^{a+j+N_j}(y_q^N - x_p^N)}{y_q - x_p\omega^{a+j+N_j}},$$

where also $n_1 + \cdots + n_L \equiv 0 \, (\text{mod}\, N)$, as is required by the periodic boundary conditions.

We have used $\prod_{\ell=1}^{N}(y - x\omega^\ell) = y^N - x^N$. Particularly, when $n_j \equiv 0$, we find $N_j = 0$ and

$$\langle\Omega|\mathcal{T}_Q(x_q, y_q)|\Omega\rangle = N^{1-\frac{1}{2}L}y_p^{rN}(x_q/y_p)^Q\mathcal{P}_Q(x_q/y_p),$$

while for $n_j \equiv N - 1$, we substitute $N_j = (j-1)(N-1)$ to obtain

$$\langle\bar{\Omega}|\mathcal{T}_Q(x_q, y_q)|\Omega\rangle = N^{1-\frac{1}{2}L}\delta_{Q,0}\omega^{-L(L+1)/2}(y_p^N - x_q^N)^r.$$

Similarly we find

$$\langle\bar{\Omega}|\mathcal{T}_Q(x_q, y_q)|\bar{\Omega}\rangle = N^{1-\frac{1}{2}L}\omega^Q(\mu_p y_q/\mu_q)^{rN}(x_p/y_q)^Q\mathcal{P}_Q(x_p/y_q)$$

and

$$\langle\Omega|\mathcal{T}_Q(x_q, y_q)|\bar{\Omega}\rangle = N^{1-\frac{1}{2}L}\delta_{Q,0}\omega^{L(L+1)/2}(y_p^N - x_q^N)^r.$$

Since L is a multiple of N, the right-hand-sides of the 2nd and 4th equations are equal, meaning

$$\mathcal{T}_0(x_q, y_q) = \prod_{j=1}^{r}[X_j - \mathbf{H}_j Y_j + (\mathbf{E}_j^+ + \mathbf{E}_j^-)Z_j],$$

with equality of the coefficients of $\mathbf{E}_j^+$ and $\mathbf{E}_j^-$.

Using methods described by Deguchi, we can derive by induction

$$(\mathbf{x}_0^+)^{(n-1)}(\mathbf{x}_1^-)^{(n)}|\Omega\rangle = \sum_{j=1}^{n} \Lambda_{n-j}\,\mathbf{x}_j^-\,|\Omega\rangle,$$

$$(\mathbf{x}_1^-)^{(n-1)}(\mathbf{x}_0^+)^{(n)}|\bar{\Omega}\rangle = \sum_{j=1}^{n} \Lambda_{n-j}\,\mathbf{x}_{j-1}^+\,|\bar{\Omega}\rangle,$$

$$\langle\Omega|(\mathbf{x}_0^+)^{(n)}(\mathbf{x}_1^-)^{(n-1)} = \sum_{j=1}^{n} \Lambda_{n-j}\langle\Omega|\mathbf{x}_{j-1}^+,$$

$$\langle\bar{\Omega}|(\mathbf{x}_1^-)^{(n)}(\mathbf{x}_0^+)^{(n-1)} = \sum_{j=1}^{n} \Lambda_{n-j}\langle\bar{\Omega}|\mathbf{x}_j^-,$$

which we can invert using

$$\sum_{n=0}^{m} \Lambda_{m-n}S_n = \delta_{m,0}, \quad \text{with} \quad S_0 = 1, \quad \Lambda_0 = 1, \quad S_n = \sum_{i=1}^{r} z_i^{-n}\beta_{i,0},$$

for $1-r < n < r-1$, giving

$$\mathbf{x}_j^- |\Omega\rangle = \sum_{n=1}^{j} S_{j-n} (\mathbf{x}_0^+)^{(n-1)} (\mathbf{x}_1^-)^{(n)} |\Omega\rangle,$$

$$\mathbf{x}_{j-1}^+ |\bar{\Omega}\rangle = \sum_{n=1}^{j} S_{j-n} (\mathbf{x}_1^-)^{(n-1)} (\mathbf{x}_0^+)^{(n)} |\bar{\Omega}\rangle,$$

$$\langle\Omega| \mathbf{x}_{j-1}^+ = \sum_{n=1}^{j} S_{j-n} \langle\Omega| (\mathbf{x}_0^+)^{(n)} (\mathbf{x}_1^-)^{(n-1)},$$

$$\langle\bar{\Omega}| \mathbf{x}_j^- = \sum_{n=1}^{j} S_{j-n} \langle\bar{\Omega}| (\mathbf{x}_1^-)^{(n)} (\mathbf{x}_0^+)^{(n-1)},$$

where the more complicated looking right-hand sides are in fact more easy to handle.

Next, after obvious substitutions we arrive at

$$\langle\Omega|\mathbf{E}_m^- = -\beta_{m,0}\sum_{\ell=1}^r z_m^{\ell-1}\langle\Omega|(\mathbf{x}_0^+)^{(\ell)}(\mathbf{x}_1^-)^{(\ell-1)},$$

$$\mathbf{E}_m^-|\bar\Omega\rangle = -\beta_{m,0}\sum_{\ell=1}^r z_m^{\ell-1}(\mathbf{x}_1^-)^{(\ell-1)}(\mathbf{x}_0^+)^{(\ell)}|\bar\Omega\rangle,$$

$$\mathbf{E}_m^+|\Omega\rangle = \beta_{m,0}\sum_{\ell=1}^r z_m^{\ell}(\mathbf{x}_0^+)^{(\ell-1)}(\mathbf{x}_1^-)^{(\ell)}|\Omega\rangle,$$

$$\langle\bar\Omega|\mathbf{E}_m^+ = \beta_{m,0}\sum_{\ell=1}^r z_m^{\ell}\langle\bar\Omega|(\mathbf{x}_1^-)^{(\ell)}(\mathbf{x}_0^+)^{(\ell-1)}.$$

Using

$$\frac{\mathbf{e}^n}{[n]!}|n'\rangle = \begin{bmatrix} n' \\ n \end{bmatrix}|n'-n\rangle, \quad \frac{\mathbf{f}^n}{[n]!}|n'\rangle = \begin{bmatrix} n'+n \\ n \end{bmatrix}|n'+n\rangle,$$

$$\langle n'|\frac{\mathbf{e}^n}{[n]!} = \begin{bmatrix} n'+n \\ n \end{bmatrix}\langle n'+n|, \quad \langle n'|\frac{\mathbf{f}^n}{[n]!} = \begin{bmatrix} n' \\ n \end{bmatrix}\langle n'-n|.$$

we obtain

$$(\mathbf{x}_1^-)^{(\ell)}|\Omega\rangle = \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \cdots + n_L = \ell N}} \omega^{-\sum_j jn_j}|\{n_j\}\rangle,$$

$$(\mathbf{x}_0^+)^{(\ell)}|\bar{\Omega}\rangle = (-1)^\ell \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \cdots + n_L = \ell N}} |\{N-1-n_j\}\rangle,$$

$$\langle\Omega|(\mathbf{x}_0^+)^{(\ell)} = \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \cdots + n_L = \ell N}} \omega^{\sum_j jn_j}\langle\{n_j\}|,$$

$$\langle\bar{\Omega}|(\mathbf{x}_1^-)^{(\ell)} = (-1)^\ell \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \cdots + n_L = \ell N}} \langle\{N-1-n_j\}|,$$

and

$$(\mathbf{x}_0^+)^{(\ell)}(\mathbf{x}_1^-)^{(\ell+1)}|\Omega\rangle = \sum_{\substack{\{0 \le n_j \le N-1\} \\ n_1+\cdots+n_L=N}} \omega^{-\sum_j j n_j} K_{\ell N}(\{n_j\})|\{n_j\}\rangle,$$

$$\langle\Omega|(\mathbf{x}_0^+)^{(\ell+1)}(\mathbf{x}_1^-)^{(\ell)} = \sum_{\substack{\{0 \le n_j \le N-1\} \\ n_1+\cdots+n_L=N}} \langle\{n_j\}|\omega^{\sum_j j n_j} \bar{K}_{\ell N}(\{n_j\}),$$

$$(\mathbf{x}_1^-)^{(\ell)}(\mathbf{x}_0^+)^{(\ell+1)}|\bar{\Omega}\rangle = - \sum_{\substack{\{0 \le n_j \le N-1\} \\ n_1+\cdots+n_L=N}} K_{\ell N}(\{n_j\})|\{N-1-n_j\}\rangle,$$

$$\langle\bar{\Omega}|(\mathbf{x}_1^-)^{(\ell+1)}(\mathbf{x}_0^+)^{(\ell)} = - \sum_{\substack{\{0 \le n_j \le N-1\} \\ n_1+\cdots+n_L=N}} \langle\{N-1-n_j\}|\bar{K}_{\ell N}(\{n_j\}),$$

where

$$K_m(\{n_j\}) \equiv \sum_{\substack{\{0 \le n'_j \le N-1\} \\ n'_1 + \cdots + n'_L = m}} \prod_{j=1}^{L} \begin{bmatrix} n_j + n'_j \\ n'_j \end{bmatrix} \omega^{n'_j N_j}, \quad N_j \equiv \sum_{\ell=1}^{j-1} n_\ell,$$

$$\bar{K}_m(\{n_j\}) \equiv \sum_{\substack{\{0 \le n'_j \le N-1\} \\ n'_1 + \cdots + n'_L = m}} \prod_{j=1}^{L} \begin{bmatrix} n_j + n'_j \\ n'_j \end{bmatrix} \omega^{n'_j \bar{N}_j}, \quad \bar{N}_j \equiv \sum_{\ell=j+1}^{L} n_\ell,$$

for integers $m \le (N-1)L$. The generating function of $K_m(\{n_j\})$ for $\sum n_j = kN$ is

$$g(\{n_j\}, t) \equiv \sum_{m=0}^{(N-1)L-kN} K_m(\{n_j\}) t^m = \frac{1}{(1-t^N)^k} \prod_{j=1}^{L} \frac{1-t^N}{1-t\omega^{N_j}}.$$

We define the following polynomials for $n_1 + \cdots + n_L = N$,

$$G_Q(\{n_j\}, z) \equiv \sum_{\ell=0}^{m_Q-1} K_{Q+\ell N}(\{n_i\}) z^\ell = \frac{t^{-Q}}{N(1-t^N)} \sum_{a=0}^{N-1} \omega^{-Qa} \prod_{j=1}^{L} \frac{1-t^N}{1-t\omega^{a+N_j}},$$

and

$$\bar{G}_Q(\{n_j\}, z) \equiv \sum_{\ell=0}^{m_Q-1} \bar{K}_{Q+\ell N}(\{n_j\}) z^\ell = \frac{t^{-Q}}{N(1-t^N)} \sum_{a=0}^{N-1} \omega^{-Qa} \prod_{j=1}^{L} \frac{1-t^N}{1-t\omega^{a+\bar{N}_j}},$$

where $z = t^N$. Here the right-hand equalities are proved using the generating functions g (and $\bar{g}$ which is similarly defined).

We then have

$$\langle\Omega|\mathbf{E}_m^- = -\beta_{m,0} \sum_{\substack{\{0\leq n_j\leq N-1\} \\ n_1+\cdots+n_L=N}} \langle\{n_j\}|\, \omega^{\sum_j jn_j} \bar{G}(\{n_j\},z_m),$$

$$\langle\bar{\Omega}|\mathbf{E}_m^+ = -\beta_{m,0}z_m \sum_{\substack{\{0\leq n_j\leq N-1\} \\ n_1+\cdots+n_L=N}} \langle\{N-1-n_j\}|\, \bar{G}(\{n_j\},z_m),$$

$$\mathbf{E}_k^+|\Omega\rangle = \beta_{k,0}z_k \sum_{\substack{\{0\leq n_j\leq N-1\} \\ n_1+\cdots+n_L=N}} \omega^{-\sum_j jn_j} G(\{n_j\},z_k)|\{n_j\}\rangle,$$

$$\mathbf{E}_k^-|\bar{\Omega}\rangle = \beta_{k,0} \sum_{\substack{\{0\leq n_j\leq N-1\} \\ n_1+\cdots+n_L=N}} G(\{n_j\},z_k)|\{N-1-n_j\}\rangle.$$

With these results we have enough machinery to determine the coefficients X_j, Y_j, and Z_j in the transfer matrices

$$\mathcal{T}_0(x_q,y_q) = \prod_{j=1}^{r}[X_j - \mathbf{H}_j Y_j + (\mathbf{E}_j^+ + \mathbf{E}_j^-)Z_j].$$

Since
$$\langle\Omega|\mathbf{E}_m^-\mathbf{E}_k^+|\Omega\rangle = \delta_{m,k},$$
we conclude that
$$\beta_{m,0}\beta_{k,0}z_k \sum_{\substack{\{0\leq n_j\leq N-1\}\\ n_1+\cdots+n_L=N}} \bar{G}(\{n_j\},z_m)G(\{n_j\},z_k) = -\delta_{mk}\,.$$
Next, we introduce the polynomials
$$\mathsf{h}_k(z) \equiv \sum_{\substack{\{0\leq n_j\leq N-1\}\\ n_1+\cdots+n_L=N}} \bar{G}(\{n_j\},z_k)G(\{n_j\},z),$$
$$\bar{\mathsf{h}}_k(z) \equiv \sum_{\substack{\{0\leq n_j\leq N-1\}\\ n_1+\cdots+n_L=N}} \bar{G}(\{n_j\},z)G(\{n_j\},z_k).$$
With the degree of polynomial $G(\{n_j\},z)$, the degree of $\mathsf{h}_k(z)$ is also $r-1$. From the above orthogonality we know its $r-1$ roots. Hence,
$$\mathsf{h}_k(z) = -f_k(z)/z_k\beta_{k,0}^2 = \beta_{k,0}^{-1}\prod_{\ell\neq k}(z-z_\ell) = \bar{\mathsf{h}}_k(z).$$

Using what we have so far:

$$\langle\Omega|\mathbf{E}_m^-\mathcal{T}_0(x_q,y_q)|\Omega\rangle = -y_p^{rN}(1-x_q^N/y_p^N)N^{1-\frac{1}{2}L}\prod_{\ell\neq m}(x_q^N/y_p^N-z_\ell).$$

$$\langle\bar{\Omega}|\mathbf{E}_m^+\mathcal{T}_0(x_q,y_q)|\bar{\Omega}\rangle = -z_m(\mu_p x_p/\mu_q)^{rN}(1-y_q^N/x_p^N)N^{1-\frac{1}{2}L}\prod_{\ell\neq m}(y_q^N/x_p^N-z_\ell),$$

$$\langle\bar{\Omega}|\mathcal{T}_Q(x_q,y_q)|\Omega\rangle = N^{1-\frac{1}{2}L}\delta_{Q,0}\omega^{-L(L+1)/2}(y_p^N-x_q^N)^r,$$

$$\langle\Omega|\mathcal{T}_Q(x_q,y_q)|\bar{\Omega}\rangle = N^{1-\frac{1}{2}L}\delta_{Q,0}\omega^{L(L+1)/2}(y_p^N-x_q^N)^r,$$

and similar relations for $\hat{\mathcal{T}}_0(x_q,y_q)$.

These give enough equation to determine the "Onsager rotations" leading to what we gave before, solving the $Q=0$ ground state sector.

The identities for polynomials and generating functions based on roots of Drinfeld polynomials can be generalized to $Q\neq 0$ cases, to twisted boundary conditions, and to N not a multiple of L. Part of that is done in our third paper.

Quantum loop subalgebra for $Q \neq 0$

We must recall the τ_2 transfer matrix for spin shift parameter Q:

$$\tau_2(t_q)|_Q = \mathbf{A}(t_q) + \omega^{-Q} \mathbf{D}(t_q),$$

the monodromy operator

$$\mathsf{U}(t_q) = \sum_{j=0}^{L} (-\omega t_q/t_p)^j \begin{pmatrix} \mathbf{A}_j & \mathbf{B}_j \\ \mathbf{C}_j & \mathbf{D}_j \end{pmatrix}.$$

and the definitions

$$\mathbf{B}_1^{(n)} = \frac{(\mathbf{B}_1)^n}{[n]!}, \quad \mathbf{B}_L^{(n)} = \frac{(\mathbf{B}_L)^n}{[n]!},$$

$$\mathbf{C}_0^{(n)} = \frac{(\mathbf{C}_0)^n}{[n]!}, \quad \mathbf{C}_{L-1}^{(n)} = \frac{(\mathbf{C}_{L-1})^n}{[n]!},$$

(with a limiting process $q = r\omega$, $r \uparrow 1$ assumed if $n \geq N$).

For $Q \neq 0$ the special states $|\Omega\rangle$ and $|\bar{\Omega}\rangle$ have different eigenvalues:

$$\tau_2(t_q)|_Q \, |\Omega\rangle = \left[(1 - \omega t)^L + \omega^{-Q}(1-t)^L \right] |\Omega\rangle,$$
$$\tau_2(t_q)|_Q \, |\bar{\Omega}\rangle = \left[\omega^{-Q}(1 - \omega t)^L + (1-t)^L \right] |\bar{\Omega}\rangle.$$

We find that

$$\prod_{j=1}^{J} \mathbf{C}_0^{(m_j N + Q)} \mathbf{B}_1^{(n_j N + Q)} |\Omega\rangle, \quad \prod_{j=1}^{J} \mathbf{C}_{L-1}^{(m_j N + N - Q)} \mathbf{B}_L^{(n_j N + N - Q)} |\Omega\rangle$$

are eigenvectors in the same degenerate eigenspace as $|\Omega\rangle$, while

$$\prod_{j=1}^{J} \mathbf{B}_1^{(m_j N + N - Q)} \mathbf{C}_0^{(n_j N + N - Q)} |\bar{\Omega}\rangle, \quad \prod_{j=1}^{J} \mathbf{B}_L^{(m_j N + Q)} \mathbf{C}_{L-1}^{(n_j N + Q)} |\bar{\Omega}\rangle$$

are eigenvectors in the same degenerate eigenspace as $|\bar{\Omega}\rangle$.

For $Q \neq 0$, these two eigenspaces both have dimension 2^{r-1}. However, for $Q = 0$ they merge to one eigenspace of dimension 2^r.

Defining

$$\mathbf{x}_{1,Q}^{-} = \frac{\omega^Q}{\Lambda_0^Q (1-\omega)^{N+2Q}} \, \mathbf{C}_0^{(Q)} \mathbf{B}_1^{(N+Q)},$$

$$\mathbf{x}_{0,Q}^{+} = \frac{\omega^Q}{\Lambda_0^Q (1-\omega)^{N+2Q}} \, \mathbf{C}_0^{(N+Q)} \mathbf{B}_1^{(Q)},$$

we can construct the loop subalgebra generated by

$$\mathbf{h}_{1,Q} = [\mathbf{x}_{0,Q}^{+}, \mathbf{x}_{1,Q}^{-}],$$

$$\mathbf{x}_{n+2,Q}^{-} = \tfrac{1}{2}[\mathbf{h}_{1,Q}, \mathbf{x}_{n+1,Q}^{-}],$$

$$\mathbf{x}_{n+1,Q}^{+} = -\tfrac{1}{2}[\mathbf{h}_{1,Q}, \mathbf{x}_{n,Q}^{+}],$$

$$\mathbf{h}_{n+1,Q} = [\mathbf{x}_{n,Q}^{+}, \mathbf{x}_{1,Q}^{-}],$$

for $0 \le n \le \infty$, provided we can prove the Serre relations

$$[[[\mathbf{x}_{0,Q}^{+}, \mathbf{x}_{1,Q}^{-}], \mathbf{x}_{1,Q}^{-}], \mathbf{x}_{1,Q}^{-}] = 0, \qquad [\mathbf{x}_{0,Q}^{+}, [\mathbf{x}_{0,Q}^{+}, [\mathbf{x}_{0,Q}^{+}, \mathbf{x}_{1,Q}^{-}]]] = 0.$$

For now, this is a conjecture backed up by some special cases proved analytically or checked by Maple 12.

We can then introduce

$$\mathbf{E}^+_{m,Q} = \sum_{n=0}^{r-1} \beta^Q_{m^*,n} z_{m,Q} \mathbf{x}^-_{n+1,Q}, \quad \mathbf{E}^-_{m,Q} = -\sum_{n=0}^{r-1} \beta^Q_{m^*,n} \mathbf{x}^+_{n,Q},$$

$$\mathbf{H}_{m,Q} = [\mathbf{E}^+_{m,Q}, \mathbf{E}^-_{m,Q}] = \sum_{n=0}^{r-1} \beta^Q_{m^*,n} z_{m,Q} \mathbf{h}_{n+1,Q},$$

for $m = 1, \ldots, r-1$, and with $\beta^Q_{m,n}$ and $z_{m,Q}$ derived from the Drinfeld polynomial for $Q \neq 0$ given before.

The 2^{r-1} eigenvectors of the superintegrable chiral Potts transfer matrix can then all be given, again in terms of rotation matrices $\mathcal{R}$ and $\mathcal{S}$.

Similarly, another set of 2^{r-1} eigenvectors can be constructed starting with $|\bar{\Omega}\rangle$, instead of $|\Omega\rangle$.

Summary

- The integrable chiral Potts model is a special parafermionic model. To better understand this we need to evaluate correlation functions. From Baxter's Z-invariance, it is enough to do this for the superintegrable subcase.

- We have constructed 2^r eigenvectors of the transfer matrix for $Q = 0$ starting from the τ_2 model and finding the loop group and the rotation matrices to be used.

- We have also constructed two sets of 2^{r-1} eigenvectors of the transfer matrix for all $Q \neq 0$.

- Having the eigenvectors in the ground state sectors is a major step towards a better understanding of the order parameters and correlation functions. Helen Au-Yang will proceed from the results discussed here and give a derivation of the order parameters that were conjectured in 1988 by Albertini, McCoy, Perk, and Tang, and first proved by Baxter in 2005 using functional equations.

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Further references are provided in Helen Au-Yang's talks .