Integrable Chiral Potts Model

Its history and relation with Mathematics

Helen Au-Yang and Jacques Perk

Oklahoma State University (on leave)
Australian National University

Feb. 15, 2009

Its Beginning

Chiral Clock Model:

Hamiltonian Limit : Lax Pair:

High Genus Solutions of the Star-Triangle Equation:

Relation with Quantum Group

Quantum Group $\hat{U}_q(\mathfrak{sl}_2)$ and Chiral Potts Model

Functional Relation

Integrable Chiral Potts Model

Critical Exponents

Monte Carlo Simulations

Basic Hypergeometric Series

Saalschützian and Star-Triangle Relation

Two sided Series of Gamma Functions

Chiral Clock Model [Howes&Kadanoff 1983],

- Three State Chiral Clock Model: [HSF 1982, Fisher 1984]

$$-\beta\mathcal{E} = \sum_{i,j}[K_{ij} + \bar{K}_{ij}], \quad K_{ij} = K\cos\left(\frac{2\pi}{3}(n_{i,j} - n_{i+1,j} + \Delta)\right),$$

$$\bar{K}_{ij} = \bar{K}\cos\left(\frac{2\pi}{3}(n_{i,j} - n_{i,j+1} + \bar{\Delta})\right) = \bar{K}(n_{i,j} - n_{i,j+1})$$

  where $n_{i,j} = 0, 1, 2$ is the variable associated with site (i, j) of a square lattice and $(\Delta, \bar{\Delta})$ is the chiral field.

- Commensurate-Incommensurate Phase Transitions:

  $\Delta = \bar{\Delta} = 0$: $K(0) > K(1) = K(-1)$

  ferromagnetic Potts Model; (nondegenerate ground state);

  $\Delta = \bar{\Delta} = 3/2$: $K(0) < K(1) = K(-1)$

  anti-ferromagnetic Potts (infinitely degenerate ground state)
  By varying Δ, the system must undergo a phase transition.

- Chiral: $2\Delta \neq 0 (\mathrm{mod}\ 3) \rightarrow K(n) \neq K(-n)$

CONSERVATION LAWS FOR Z(N) SYMMETRIC QUANTUM SPIN MODELS AND THEIR EXACT GROUND STATE ENERGIES

Francisco C. ALCARAZ*

Department of Mathematics, The Faculties, Australian National University, Canberra ACT 2600 Australia

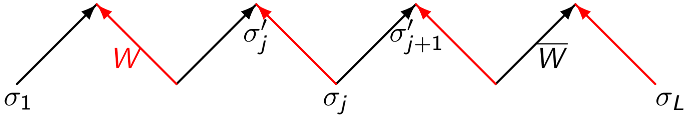
A. LIMA SANTOS

Departamento de Fisica, Universidade Estadual Paulista

ACKNOWLEDGEMENT We would like to thank M.N. Barber, M. Batchelor, B. Davies, G. Newsam, P. Pearce, S. Roberts and P. Rujan for relevant discussions and T. Lynam and A. Zalucki for help with the manuscript. This work was supported in part by the Australian Research Grant Scheme and by Fundação de Amparo à Pesquisa do Estado de São Paulo, Brasil.

Local Operators

- Transfer Matrix $\mathcal{T}_{\sigma,\sigma'} = \prod_{j=1}^{L} W(\sigma_j, \sigma'_j)\bar{W}(\sigma_j, \sigma'_{j+1})$



where $\sigma_j = \omega^{n_j}$, $n_j = 0, \cdots, N-1$ and $\omega^N = 1$;
$W(\sigma, \sigma') = \mathrm{e}^{-\beta\mathcal{E}(\sigma,\sigma')}$; $\overline{W}(\sigma, \sigma') = \mathrm{e}^{-\beta\overline{\mathcal{E}}(\sigma,\sigma')}$

- Edge Operators

$$\mathbf{L}_k(u) = \sum_{j=0}^{N-1} \ell_j(u)\rho_k^j, \quad \ell_j(u) = \frac{W(j)}{W(0)}, \quad \rho_k = \mathbf{X}_k$$

$$\mathbf{L}_{k+\frac{1}{2}}(u) = \sum_{j=0}^{N-1} \bar{\ell}_j(u)\rho_{k+\frac{1}{2}}^j, \quad \bar{\ell}_j(u) = \frac{\overline{W}^{(f)}(j)}{\overline{W}^{(f)}(0)}, \quad \rho_{k+\frac{1}{2}} = \mathbf{Z}_k\mathbf{Z}_{k+1}^\dagger$$

- Weyl Operators

$$\mathbf{Z}_{m,n} = \delta_{m,n}\omega^n, \quad \mathbf{X}_{m,n} = \delta_{m,n+1}, \quad \mathbf{ZX} = \omega\mathbf{XZ}$$
$$\mathbf{X}_k = \mathbf{1} \otimes \cdots \otimes \mathbf{X} \otimes \cdots \otimes \mathbf{1}, \; \mathbf{Z}_k = \mathbf{1} \otimes \cdots \otimes \mathbf{Z} \otimes \cdots \otimes \mathbf{1}$$

Commuting Transfer Matrices

- Star-Triangle Equation in Operator Form

$$\mathbf{M}_k(u,u')=\sum_{j=0}^{N-1} x_j(u,u')\rho_k^j, \quad \mathbf{M}_{k+\frac{1}{2}}(u,u')=\sum_{j=0}^{N-1} \bar{x}_j(u,u')\rho_{k+\frac{1}{2}}^j$$

$$\mathbf{M}_{k-\frac{1}{2}}(u,u')\mathbf{L}_k(u)\mathbf{L}_{k-\frac{1}{2}}(u') = \mathbf{L}_{k-\frac{1}{2}}(u')\mathbf{L}_k(u)\mathbf{M}_k(u,u')$$

- Condition for integrability

$$\frac{V_{\alpha\beta}V_{00}}{V_{\alpha 0}V_{0\beta}} = \frac{\bar{V}_{\beta\alpha}\bar{V}_{00}}{\bar{V}_{\beta 0}\bar{V}_{0\alpha}}, \quad V_{\alpha\beta} = \sum_{m=0}^{N-1}\sum_{k=0}^{N-1} \omega^{\alpha m+\beta K+mk}\ell_m \bar{\ell}'_k$$

Lax-Pair: Hamiltonian Limit

Hamiltonian Limit [AlcSa 1986], [BaPok 1980]

$$\mathcal{T} = \mathbf{1} + u\mathcal{H} + O(u^2) \quad [\mathcal{T}, \mathcal{H}] = 0$$

$$\ell'_j(u) = 1 + \alpha_j + O(u^2), \quad \bar{\ell}'_j(u) = 1 + \bar{\alpha}_j + O(u^2)$$

Integrable condition becomes linear in α and $\bar{\alpha}$

High Genus Solutions:

- Self-Dual $\ell_n = \bar{\ell}_n$: N=3 : genus=1
- Non-Self-Dual $\ell_n \neq \bar{\ell}_n$: N=3 : genus=10[AMPTY,1987]
- Self-Dual : N=3,4,5 : [MPTS,1987]

$$\ell_n = \bar{\ell}_n = \prod_{j=1}^{N-1} \frac{\omega x_1 - x_2 \omega^j}{x_4 - x_3 \omega^j}, \quad x_1^N + x_3^N = x_2^N + x_4^N$$

Star-Triangle Equations [BPA,1988]

$$\sum_{d=1}^{N} \overline{W}_{qr}(b-d)\, W_{pr}(a-d)\, \overline{W}_{pq}(d-c) = R_{pqr}\, W_{pq}(a-b)\, \overline{W}_{pr}(b-c)\, W_{qr}(a-c)$$

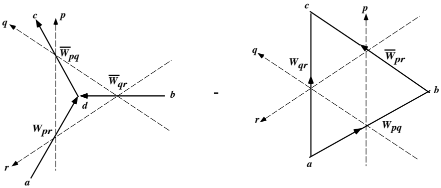


Figure : **The Star-Triangle Relations, which allow one to move a rapidity line p through a vertex.**

$$W_{pq}(n) = \left(\frac{\mu_p}{\mu_q}\right)^n \prod_{j=1}^{n} \frac{y_q - x_p\omega^j}{y_p - x_q\omega^j},$$

$$\overline{W}_{pq}(n) = \left(\mu_p\mu_q\right)^n \prod_{j=1}^{n} \frac{\omega x_p - x_q\omega^j}{y_q - y_p\omega^j}$$

$$p \to (x_p, y_p, \mu_p)$$

$$x_p^N + y_p^N = k(1 + x_p^N y_p^N)$$

$$k'^2 + k^2 = 1, \quad k'\mu_p^N = 1 - k y_p^N$$

Genus

[Sah-Kuga]

$$x^N + y^N = k(1 + x^N y^N)$$
$$\mu^N = k'/(1 - kx^N)$$

$$(N-1)(N^2 - N - 1)$$
g=1,10,33,76 for N=2,3,4,5

[Davies and Neeman]

$$x^N + y^N = k(1 + x^N y^N)$$

$$(N-1)^2$$
g=1, 4, 9, 16 for N=2, 3, 4, 5

[Self-Dual, Sah]

$$x^N + y^N = 1$$

$$\frac{1}{2}(N-1)(N-2)$$
g=0, 1, 3, 6 for N=2, 3, 4, 5

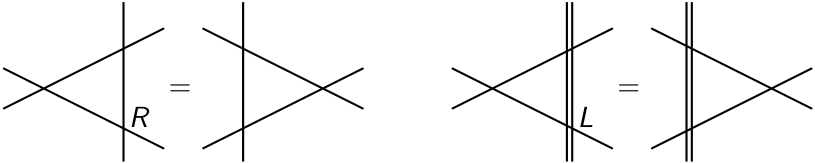
[Fateev-Zamolodichov]

$$x^N + y^N = 0$$

0

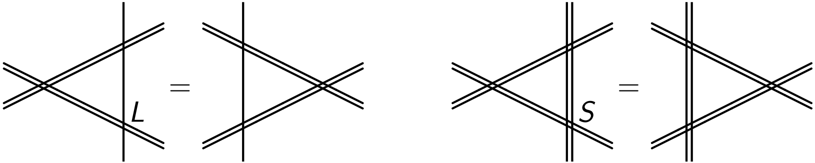
The Bazhanov–Stroganov construction; [BS 1990]

Yang Baxter Equations



R : $2 \otimes 2$ Highest weight representation: $\hat{U}_q(\mathfrak{sl}_2)$; [Jimbo, Drinfeld]
L : $2 \otimes N$: Cyclic representations of the $\hat{U}_q(\mathfrak{sl}_2)$; [Jimbo]

Intertwiners

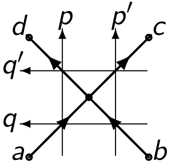


Chiral Potts Model S: $N \otimes N$ is the intertwiner of the Cyclic Representations. $\rightarrow$ Yang-Baxter equation.

Functional Relation

[Baxter,Bazhanov,Perk,1990]

Square $U(a,b,c,d)$



Let $(x_{q'}, y_{q'}, \mu_{q'}) = (y_q, \omega^j x_q, \mu_q^{-1})$

If $0 \le a-d \le j-1$
$U^{(j)}(a,b,c,d) = 0$, for $j \le b-c \le N-1$

$$\tau_j(t_q) = tr[\prod_{n=1}^{L} U^{(j)}(\sigma_n, \sigma_{n+1}, \sigma'_{n+1}, \sigma'_n)]$$

Functional Equations

$$T_q\, \hat{T}_{q'} = B^{(j)}_{pp'q}\, \mathbf{X}^{-j}\, \tau_j(t_q) + \bar{B}^{(N-j)}_{pp'q}\, \tau_{N-j}(\omega^j t_q) \qquad t_q \equiv x_q y_q$$

$$\tau_j(t_q)\, \tau_2(\omega^{j-1} t_q) = z(\omega^{j-1} t_q)\, \mathbf{X}\, \tau_{j-1}(t_q) + \tau_{j+1}(t_q)$$

$$\tau_0(t) = 0, \quad \tau_1(t) = \mathbf{1}, \quad \tau_{N+1}(t_q) = z(t_q)\, \mathbf{X}\, \tau_{N-1}(\omega t_q) + (\alpha_q + \bar{\alpha}_q)\, \mathbf{1}$$

where the B's, $z(t)$, α_q and $\bar{\alpha}_q$ are known scalar functions.

Critical Exponents and Scaling Hypothesis

- Specific Heat

  $$C_v \to t^{-\alpha}, \quad t = 1 - T/T_c$$

- Order Parameter
  $\langle \sigma \rangle \to t^{\beta}$
- Susceptibiblity $\chi \to t^{-\gamma}$
- Correlation Length

  $$\langle \sigma_0 \sigma_{\mathbf{r}} \rangle_c \to e^{-\mathbf{r}/\xi}, \quad \xi \to t^{-\nu}$$

- Interfacial tension $\epsilon \to t^{\mu}$
- Scaling Relation

  $$d\nu = 2 - \alpha, \quad \nu = \mu$$

- [Baxter 1989] Free energy
  $\alpha = 1 - \frac{2}{N}$
- [AMPT 1989, Baxter 2005]
  $\beta_n = \frac{n(N-n)}{2N^2}$
- $d\nu = 2 - \alpha$
  $d = 2 \quad \nu = \frac{1}{2} + \frac{1}{N}$
- [Baxter 1993]
  $\mu = \frac{1}{2} + \frac{1}{N}$
- $\nu = \mu = \frac{1}{2} + \frac{1}{N}$

Phase diagram of 3-state symmetric model

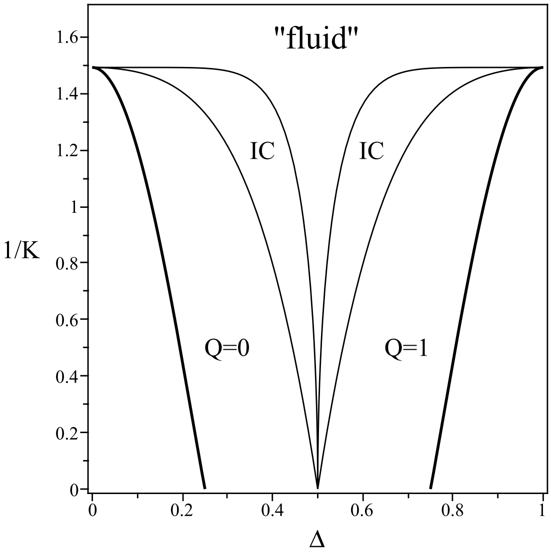


Figure : **The three state chiral Clock Model.**

- $Q = 0$: Ferromagnetic phase;
- $Q = 1$: Commensurate phase;
- IC: incommensurate phase
- fluid : disordered phase

Cyclic Hypergeometric Functions

Basic hypergeometric hypergeometric series

$$ {}_{p+1}\Phi_p \left[\begin{matrix} \alpha_1, \cdots, \alpha_{p+1} \\ \beta_1, \cdots, \beta_p \end{matrix} ; z \right] = \sum_{l=0}^{\infty} \frac{(\alpha_1; q)_l \cdots (\alpha_{p+1}; q)_l}{(\beta_1; q)_l \cdots (\beta_p; q)_l (q; q)_l} z^l, $$

with q-Pochhammer symbol $(x; q)_l \equiv \prod_{j=1}^{l} (1 - xq^{j-1})$.

Cyclic hypergeometric function with summand periodic mod N

Setting $\alpha_{p+1} = q^{1-N}$ first and then $q \to \omega \equiv e^{2\pi \mathrm{i}/N} \to$ finite sum

$$ {}_{p+1}\Phi_p \left[\begin{matrix} \omega, \alpha_1, \cdots, \alpha_p \\ \beta_1, \cdots, \beta_p \end{matrix} ; z \right] = \sum_{l=0}^{N-1} \frac{(\alpha_1; \omega)_l \cdots (\alpha_p; \omega)_l}{(\beta_1; \omega)_l \cdots (\beta_p; \omega)_l} z^l $$

With the periodicity requirement

$$ z^N = \prod_{j=1}^{p} \gamma_j^N, \qquad {\gamma_j}^N = \frac{1 - \beta_j^N}{1 - \alpha_j^N}, $$

Application to the Integrable Chiral Potts Model

The weights of the integrable chiral Potts model can be written in product form

$$W(n) = \gamma^n \frac{(\alpha;\omega)_n}{(\beta;\omega)_n}, \qquad \gamma^N = \frac{1-\beta^N}{1-\alpha^N}, \qquad (W(0)=1).$$

The dual weights are given by the discrete Fourier transform

$$W^{(\mathrm{f})}(k) = \sum_{n=0}^{N-1} \omega^{nk}\, W(n) = {}_2\Phi_1\left[\begin{matrix}\omega,\alpha\\ \beta\end{matrix};\gamma\,\omega^k\right]$$

which can be expressed by products.
If $z=\omega$, ${}_3\Phi_2$ can be similarly summed.

Under the Saalschütz condition
$\omega^2\alpha_1\alpha_2\cdots\alpha_p = \beta_1\beta_2\cdots\beta_p$ (and $z=\omega$), then ${}_4\Phi_3$ can also be summed. The resulting formula is the star-triangle equation.

$N \to \infty$ Limits

The double sided series is summable

$$\sum_{n=-\infty}^{\infty} \frac{\Gamma(x_1+n)\Gamma(x_2+n)\Gamma(x_3+n)}{\Gamma(y_1+n)\Gamma(y_2+n)\Gamma(y_3+n)} = \frac{G(x_1,x_2,x_3|y_1,y_2,y_3)}{\prod_{i=1}^{3}\prod_{j=1}^{3}\Gamma(y_i-x_j)},$$

if both the Saalschütz condition and the periodicity condition hold, i.e.

$$x_1+x_2+x_3+2 = y_1+y_2+y_3$$
$$\sin\pi x_1 \ \sin\pi x_2 \ \sin\pi x_3 = \sin\pi y_1 \ \sin\pi y_2 \ \sin\pi y_3$$

where

$$G(x_1,x_2,x_3|y_1,y_2,y_3) \equiv \prod_{j=2}^{3}\Gamma(x_j)\Gamma(1-x_j)\prod_{i=1}^{3}\Gamma(y_i-x_1)\Gamma(1-y_i+x_1)$$

- Permutations of x_1, x_2, x_3 and y_1, y_2, y_3,
- Reflections $x_j \mapsto 1-y_j,\ y_j \mapsto 1-x_j$ simultaneously,
- Translations $x_j \mapsto x_j+M,\ y_j \mapsto y_j+M$ for $j=1,2$ or 3,