Spontaneous Magnetization in the Integrable Chiral Potts Model

Two Different Approaches

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- The spontaneous magnetization is independent of the rapidities and depends only on the "universal" parameter k (or k'). Hence to calculate $\mathcal{M}_r$ for the general model it would be sufficient to calculate it for the superintegrable model.
- The "superintegrable" chiral Potts model has similarities to the Ising model (Onsager algebra).
- Question: Is there an algebraic (Ising-like) way of obtaining $\mathcal{M}_r$ for the superintegrable chiral Potts model - may give insights into the algebraic structure.

Baxter's Approach

Consider a lattice of L columns. On each site i there lives a "spin" σ_i, with values $0, 1, \ldots, N-1$.

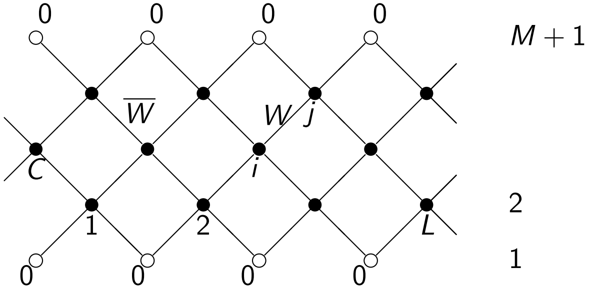

Figure: The square lattice $\mathcal{L}$ turned through 45°.

Set $\omega = \mathrm{e}^{2\pi\mathrm{i}/N}$ and define $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_L\}$ be the set of L spins on a row of the lattice. Let v_P be the N^L-dimensional vector with entries

$$(v_P)_\sigma = |P; \{0\}\rangle = N^{-1/2} \sum_{b=0}^{N-1} \omega^{-Pb}\, \delta(\sigma_1, b)\, \delta(\sigma_2, b) \cdots \delta(\sigma_L, b)$$

Hamiltonian

and let $\mathcal{Z}_1, \ldots, \mathcal{Z}_L$ and $\mathcal{X}_1, \ldots, \mathcal{X}_L$ be matrices (operators)

$$(\mathcal{Z}_j)_{\sigma,\sigma'} = \omega^{\sigma_j} \prod_{m=1}^{L} \delta(\sigma_j, \sigma_j'),$$

$$(\mathcal{X}_j)_{\sigma,\sigma'} = \delta(\sigma_j, \sigma_j' + 1) \prod_{m=1, m\neq j}^{L} \delta(\sigma_j, \sigma_j').$$

Define two hamiltonians

$$\mathcal{H}_0 = -2 \sum_{j=1}^{L} \sum_{n=1}^{N-1} \mathcal{Z}_j^n \mathcal{Z}_{j+1}^{-n} / (1 - \omega^{-n})$$

$$\mathcal{H}_1 = -2 \sum_{j=1}^{L} \sum_{n=1}^{N-1} \mathcal{X}_j^n / (1 - \omega^{-n})$$

then

$$\mathcal{H} = \mathcal{H}_0 + k' \mathcal{H}_1$$

Partition Functions

Define 'partition functions'

$$Z_P(\alpha) \;=\; v_P^\dagger \mathrm{e}^{-\alpha\mathcal{H}} v_P$$

$$W_{PQ}(\alpha,\beta) \;=\; v_P^\dagger \mathrm{e}^{-\alpha\mathcal{H}} \mathcal{S}_r \, \mathrm{e}^{-\beta\mathcal{H}} v_Q \,,$$

where $Q \;=\; P + r$

$$(\mathcal{S}_r)_{\sigma,\sigma'} \;=\; \omega^{r\sigma_1} \prod_{j=1}^{L} \delta(\sigma_j, \sigma'_j)\,, \quad \text{for } \; 1 \leq r \leq N-1$$

and

$$\mathcal{H} \;=\; \mathcal{H}_0 + k'\mathcal{H}_1$$

is the hamiltonian of the superintegrable chiral Potts model.
The spontaneous magnetization is

$$\mathcal{M}_r \;=\; \frac{W_{PQ}(\alpha,\beta)}{[Z_P(2\alpha) Z_Q(2\beta)]^{1/2}}\,.$$

$Z_P(\alpha)$ is related to the partition function with fixed-spin boundary conditions. $W_{PQ}(\alpha,\beta)$ is similar, but with an extra weight $\omega^{r\sigma_1}$ at a site C in the central row.

Progress

Reduced Space

Continued operation by $\mathcal{H}_0, \mathcal{H}_1$ on the vector v_P $(0 \leq P < N)$ generates a space V_P of much lower dimension than N^L: it is of dimension 2^m, where $m = m_P = \lfloor \frac{(N-1)L-P}{N} \rfloor$. Label the 2^m basis vectors of V_P as $\tilde{v}_s$, where

$$s = \{s_1, s_2, \ldots, s_m\}, \quad s_i = 0, 1, , \quad \kappa_s = s_1 + s_2 + \cdots + s_m$$

There are 2^m vectors $\tilde{v}_s = \tilde{v}(s_1, s_2, \ldots, s_m)$, each of dimension N^L.

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Definition of D_{PQ}

$$W_{PQ}(\alpha, \beta) \; = \; Z_P(\alpha) Z_Q(\beta) D_{PQ} \,.$$

Baxter expressed D_{PQ} in terms of the basis of the reduced space as (details see page 5 of the handout)

$$D_{PQ} \; = \; \sum_s \sum_{s'} x_1^{s_1} x_2^{s_2} \cdots x_m^{s_m} (S_r)_{s,s'} \, {x'_1}^{s'_1} \, {x'_2}^{s'_2} \cdots {x'_{m'}}^{s'_{m'}} \,.$$

Conjecture for D_{PQ}

In early 2009 Baxter conjectured a form for S_r to rewrite

$$D_{PQ} = \sum_{s} \sum_{s'} y_1^{s_1} y_2^{s_2} \cdots y_m^{s_m} \left(\frac{A_{s,s'} B_{s,s'}}{C_s D_{s'}} \right) {y'_1}^{s'_1} {y'_2}^{s'_2} \cdots {y'_{m'}}^{s'_{m'}} ,$$

the sum being restricted to s, s' such that $\kappa_s = \kappa_{s'}$; and

$$A_{s,s'} = \prod_{i \in W} \prod_{j \in V'} (c_i - c'_j) \quad , \quad B_{s,s'} = \prod_{i \in V} \prod_{j \in W'} (c_i - c'_j) ,$$

$$C_s = \prod_{i \in W} \prod_{j \in V} (c_j - c_i) \quad , \quad D_{s'} = \prod_{i \in V'} \prod_{j \in W'} (c'_j - c'_i) ,$$

in which c_i are related to roots w_i of the Drinfeld Polynomials by $c_i = (1 + w_i)/(1 - w_i)$, while V for a given set s is the set of integers i such that $s_i = 0$ and W the set such that $s_i = 1$, and V', W' are similarly defined for the set s'. Goto inner Goto hom

Calculation of S_r

The matrix S_r has the following properties:

$$v_P^\dagger \mathcal{S}_r \, v_Q \;=\; 1\,, \quad Q = P + r\,,$$

$$\mathcal{H}_0 \, \mathcal{S}_r - \mathcal{S}_r \mathcal{H}_0 \;=\; 0\,,$$

and if

$$\mathcal{C}_r \;=\; (\mathcal{H}_1 \, \mathcal{S}_r - \mathcal{S}_r \mathcal{H}_1 + 2r\mathcal{S}_r)/2N$$

then

$$(\mathcal{H}_1 \, \mathcal{C}_r - \mathcal{C}_r \mathcal{H}_1 + 2r\mathcal{C}_r)/2N = \mathcal{C}_r\,.$$

Baxter has since proved that his conjecture for S_r satisfies these relations. The above equations are linear in the elements of S_r, and for small $n, L \leq 6$ Baxter has verified numerically that the equations have a unique solution for the elements of S_r. Very recently, Iorgov *et al* have prove this result.

Determinantal Form of D_{PQ}

Baxter further conjectured in early 2009 that D_{PQ} is a determinant,

$$D_{PQ} \; = \; \det[I_m + YBY'B^T]$$

where Y and Y' are diagonal matrices,

$$(Y)_{i,j} \; = \; y_i \delta_{ij} \,, \;\; (Y')_{i,j} \; = \; y'_i \delta_{ij} \,, \;\; B_{ij} \; = \; \frac{f_i f'_j}{c_i - c'_j} \,.$$

while B is $m \times m'$. The constant f_i and f_j are chosen as

$$f_i^2 \; = \; \frac{\epsilon a_i}{b_i} \,, \;\; {f'_i}^2 \; = \; -\frac{\epsilon \, a'_i}{b'_i} \,, \;\; \epsilon = \pm 1 \,, \;\; a_i = \prod_{j=1}^{m'} (c_i - c'_j),$$

$$a'_i = \prod_{j=1}^{m} (c'_i - c_j) \,, \;\; b_i = \prod_{j=1, j \neq i}^{m} (c_i - c_j) \,, \;\; b'_i = \prod_{j=1, j \neq i}^{m'} (c'_i - c'_j),$$

so that B is orthogonal in the sense that

$$B^T B = I_{m'} \;\text{ if } m \geq m' \,, \;\; BB^T = I_m \;\text{ if } m \leq m' \,.$$

Recapitulation

- The $\mathfrak{sl}_2$ algebras [Davies, 1990]

$$\sum_{j=0}^{m_Q} \mathbf{x}_{j,Q}^{-} \Lambda_j^Q = 0 \quad \Rightarrow \quad P_Q(z) = \sum_{n=0}^{m_Q} \Lambda_n^Q z^n = \Lambda_{m_Q}^Q \prod_{j=1}^{m_Q} (z - z_{j,Q})$$

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- Define the generators $\mathbf{E}_{j,Q}^{\pm}$ and $\mathbf{H}_{j,Q}$ by

$$\mathbf{x}_{j,Q}^{-} = \sum_{\ell=1}^{m_Q} z_{\ell,Q}^{-j} \mathbf{E}_{\ell,Q}^{-}, \quad \mathbf{x}_{j,Q}^{+} = \sum_{\ell=1}^{m_Q} z_{\ell,Q}^{-j} \mathbf{E}_{\ell,Q}^{+},$$

$$\mathbf{h}_{j,Q} = \sum_{\ell=1}^{m_Q} z_{\ell,Q}^{-j} \mathbf{H}_{\ell,Q},$$

Recapitulation

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$$\sum_{j=0}^{m_Q} \mathbf{x}^-_{j,Q} \Lambda^Q_j = 0 \quad \Rightarrow \quad P_Q(z) = \sum_{n=0}^{m_Q} \Lambda^Q_n z^n = \Lambda^Q_{m_Q} \prod_{j=1}^{m_Q} (z - z_{j,Q})$$

- Define the generators $\mathbf{E}^{\pm}_{j,Q}$ and $\mathbf{H}_{j,Q}$ by

$$\mathbf{x}^-_{j,Q} = \sum_{\ell=1}^{m_Q} z^{-j}_{\ell,Q} \mathbf{E}^-_{\ell,Q}, \quad \mathbf{x}^+_{j,Q} = \sum_{\ell=1}^{m_Q} z^{-j}_{\ell,Q} \mathbf{E}^+_{\ell,Q},$$
$$\mathbf{h}_{j,Q} = \sum_{\ell=1}^{m_Q} z^{-j}_{\ell,Q} \mathbf{H}_{\ell,Q},$$

- Loop Algebra $\longrightarrow$ $\mathfrak{sl}_2$ Algebras

$$[\mathbf{E}^+_{\ell,Q}, \mathbf{E}^-_{n,Q}] = \delta_{\ell,n} \mathbf{H}_{\ell,Q}\,, \quad [\mathbf{H}_{\ell,Q}, \mathbf{E}^-_{n,Q}] = 2\delta_{\ell,n} \mathbf{E}^-_{\ell,Q},$$
$$[\mathbf{H}_{\ell,Q}, \mathbf{E}^+_{n,Q}] = -2\delta_{\ell,n} \mathbf{E}^+_{\ell,Q}.$$

Generators of $\mathfrak{sl}_2$ Algebras

We have the explicit form for these generators. We can calculate the generators $\mathbf{E}^{\pm}_{k,Q}$ acting on $v_P = |P; \{0\}\rangle$. $\mathbf{E}^{-}_{m,Q}|\{0\}\rangle = 0$, $\mathbf{H}_{m,Q}|\{0\}\rangle = -|\{0\}\rangle$, $\langle\{0\}|\mathbf{E}^{+}_{\ell,Q} = 0$, $\langle\{0\}|\mathbf{H}_{\ell,Q} = -\langle\{0\}|$

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Goto Proposition1

$$\langle\{0\}|\mathbf{E}^{-}_{\ell,Q} = -\frac{\beta^{Q}_{\ell,0}}{\Lambda^{Q}_{0}} \sum_{\substack{\{0\le n_j\le N-1\}\\ n_1+\cdots+n_L=N}} \langle\{n_j\}|\,\omega^{\sum_j j n_j}\,\bar{G}_Q(\{n_j\}, z_{\ell,Q}),$$

$$\mathbf{E}^{+}_{k,Q}|\{0\}\rangle = \frac{\beta^{Q}_{k,0} z_{k,Q}}{\Lambda^{Q}_{0}} \sum_{\substack{\{0\le n_j\le N-1\}\\ n_1+\cdots+n_L=N}} \omega^{-\sum_j j n_j}\, G_Q(\{n_j\}, z_{k,Q})|\{n_j\}\rangle.$$

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in which the polynomials are given as Goto hPQ

$$G_Q(\{n_j\}, z) = \sum_{n=0}^{m'-1} K_{nN+Q}(\{n_j\}) z^n,$$

$$\bar{G}_P(\{n_j\}, z) = \sum_{n=0}^{m-1} \bar{K}_{nN+P}(\{n_j\}) z^n.$$

Sums and Generating Functions

Sums

The coefficients of these polynomials are sums given by

$$K_\ell(\{n_j\}) = \sum_{\substack{\{0\le n'_j\le N-1\}\\ n'_1+\cdots+n'_L=\ell}} \prod_{j=1}^{L} \begin{bmatrix} n_j + n'_j \\ n'_j \end{bmatrix} \omega^{n'_j N_j}, \quad N_j = \sum_{\ell=1}^{j-1} n_\ell \,,$$

$$\bar{K}_\ell(\{n_j\}) = \sum_{\substack{\{0\le n'_j\le N-1\}\\ n'_1+\cdots+n'_L=\ell}} \prod_{j=1}^{L} \begin{bmatrix} n_j + n'_j \\ n'_j \end{bmatrix} \omega^{n'_j \bar{N}_j}, \quad \bar{N}_j = \sum_{\ell=j+1}^{L} n_\ell \,,$$

where $N_j = \sigma_1 - \sigma_j$ and $\bar{N}_j = \sigma_{j+1} - \sigma_1$.

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where $N_j = \sigma_1 - \sigma_j$ and $\bar{N}_j = \sigma_{j+1} - \sigma_1$.

Generating Function

$$g(\{n_j\},t) = \sum_{m=0}^{(N-1)L-N} K_m(\{n_j\})t^m$$

$$\bar{g}(\{n_j\},t) = \sum_{m=0}^{(N-1)L-N} \bar{K}_m(\{n_j\})t^m$$

Generating Function $\mathcal{G}(t, u)$

It was shown that $g(\{n_j\}, t)$ has simple form

$$g(\{n_j\}, t) = \bar{g}^*(\{n_j\}, t) = (1 - t^N)^{L-1} \prod_{j=1}^{L} (1 - t\omega^{N_j})^{-1},$$

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$$\mathcal{G}(t,u)=\sum_{\substack{\{0\le n_j\le N-1\}\\ n_1+\cdots+n_L=N}}\bar{g}^*(\{n_j\},t)g(\{n_j\},u)=\sum_{\ell=0}^{N(r-1)}\sum_{k=0}^{N(r-1)}\mathcal{G}_{\ell,k}t^\ell u^k$$

Goto schur where $r=m_0=(N-1)L/N$ for L multiple of N,

$$\mathcal{G}_{\ell,k}=\sum_{\substack{\{0\le n_j\le N-1\}\\ n_1+\cdots+n_L=N}}\bar{K}_\ell(\{n_j\})K_k(\{n_j\})$$

Goto hPQ Goto coeff

The Eigenvectors of Transfer Matrix

The 2^{m_Q} eigenvectors in the Reduced Space

$$|\mathcal{X}_s^Q\rangle = \prod_{j=1}^{m_Q} \mathcal{R}_{j,Q} \prod_{i\in W} \mathbf{E}_{i,Q}^+ |\{0\}\rangle\,, \quad |\mathcal{Y}_s^Q\rangle = \prod_{j=1}^{m_Q} \mathcal{S}_{j,Q} \prod_{i\in W} \mathbf{E}_{i,Q}^+ |\{0\}\rangle.$$

$$\mathcal{T}_Q(x_q, y_q)|\mathcal{X}_s^Q\rangle = \Delta_s^Q |\mathcal{Y}_s^Q\rangle\,, \quad \hat{\mathcal{T}}_Q(y_q, x_q)|\mathcal{Y}_s^Q\rangle = \Delta_s^Q |\mathcal{X}_s^Q\rangle$$

where Δ_s^Q denotes the eigenvalues of the transfer matrices.

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Matrix $\mathcal{S}_{j,Q}$

$$\mathcal{S}_{j,Q} = \tfrac{1}{2}(s_{11}^{j,Q} + s_{22}^{j,Q})\mathbf{1} + \tfrac{1}{2}(s_{11}^{j,Q} - s_{22}^{j,Q})\mathbf{H}_{j,Q} + s_{12}^{j,Q}\mathbf{E}_{j,Q}^{+} + s_{21}^{j,Q}\mathbf{E}_{j,Q}^{-},$$
$$\mathcal{S}_{j,Q}^{-1} = \tfrac{1}{2}(s_{22}^{j,Q} + s_{11}^{j,Q})\mathbf{1} + \tfrac{1}{2}(s_{22}^{j,Q} - s_{11}^{j,Q})\mathbf{H}_{j,Q} - s_{12}^{j,Q}\mathbf{E}_{j,Q}^{+} - s_{21}^{j,Q}\mathbf{E}_{j,Q}^{-}.$$

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$$\mathcal{S}_{j,Q}^{-1} = \tfrac{1}{2}(s_{22}^{j,Q} + s_{11}^{j,Q})\mathbf{1} + \tfrac{1}{2}(s_{22}^{j,Q} - s_{11}^{j,Q})\mathbf{H}_{j,Q} - s_{12}^{j,Q}\mathbf{E}_{j,Q}^{+} - s_{21}^{j,Q}\mathbf{E}_{j,Q}^{-}.$$

The eigenvector corresponding to the maximum eigenvalue is

$$|\mathcal{Y}_{max}^Q\rangle = |\mathcal{Y}_{\mathbf{0}}^Q\rangle = \prod_{j=1}^{m_Q} \mathcal{S}_{j,Q}|\{0\}\rangle\,, \quad \langle\mathcal{Y}_{max}^Q| = \langle\mathcal{Y}_{\mathbf{0}}^Q| = \langle\{0\}| \prod_{j=1}^{m_Q} \mathcal{S}_{j,Q}^{-1}.$$

Spontaneous Magnetization

$$\mathcal{M}_r^2 = \frac{1}{N} \sum_{Q=0}^{N-1} \langle \mathcal{Y}_{\mathbf{0}}^{Q} | \mathcal{Y}_{\mathbf{0}}^{Q-r} \rangle \langle \mathcal{Y}_{\mathbf{0}}^{Q-r} | \mathcal{Y}_{\mathbf{0}}^{Q} \rangle$$

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Calculation of $\langle\mathcal{Y}_{\mathbf{0}}^{Q}|\mathcal{Y}_{\mathbf{0}}^{P}\rangle$ for $Q = P + r$

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We again let $m = m_P$, $m' = m_Q$, $s_{ik}^{\prime j} = s_{ik}^{j,Q}$ and $s_{ik}^{j} = s_{ik}^{j,P}$

Spontaneous Magnetization

$$\mathcal{M}_r^2 = \frac{1}{N} \sum_{Q=0}^{N-1} \langle \mathcal{Y}_{\mathbf{0}}^{Q} | \mathcal{Y}_{\mathbf{0}}^{Q-r} \rangle \langle \mathcal{Y}_{\mathbf{0}}^{Q-r} | \mathcal{Y}_{\mathbf{0}}^{Q} \rangle$$

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$$\langle \mathcal{Y}_{\mathbf{0}}^{Q} | \mathcal{Y}_{\mathbf{0}}^{P} \rangle = \langle \{0\} | \prod_{j=1}^{m'} (s_{11}^{\prime j} - s_{21}^{\prime j} \mathbf{E}_{j,Q}^{-}) \prod_{j=1}^{m} (s_{22}^{j} + s_{12}^{j} \mathbf{E}_{j,P}^{+}) | \{0\} \rangle$$

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$$\mathcal{M}_r^2 = \frac{1}{N}\sum_{Q=0}^{N-1}\langle\mathcal{Y}_{\mathbf{0}}^{Q}|\mathcal{Y}_{\mathbf{0}}^{Q-r}\rangle\langle\mathcal{Y}_{\mathbf{0}}^{Q-r}|\mathcal{Y}_{\mathbf{0}}^{Q}\rangle$$

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$$\langle\mathcal{Y}_{\mathbf{0}}^{Q}|\mathcal{Y}_{\mathbf{0}}^{P}\rangle = \langle\{0\}|\prod_{j=1}^{m'}(s'^{j}_{11} - s'^{j}_{21}\mathbf{E}^{-}_{j,Q})\prod_{j=1}^{m}(s^{j}_{22} + s^{j}_{12}\mathbf{E}^{+}_{j,P})|\{0\}\rangle$$

$$= \prod_{j=1}^{m'}s'^{j}_{11}\prod_{j=1}^{m}s^{j}_{22}\langle\{0\}|\prod_{j=1}^{m_Q}(1 + y'_j\mathbf{E}^{-}_{j,Q})\prod_{j=1}^{m}(1 + y_j\mathbf{E}^{+}_{j,P})|\{0\}\rangle$$

where

$$y_j = s^{j}_{12}/s^{j}_{22}\,, \quad y'_j = -s'^{j}_{21}/s'^{j}_{11}.$$

Expansion of the Products

We expand the product, and since

$$\Big(\prod_{s=1}^{k} \mathbf{E}^{+}_{\ell_s,P}\Big)|\{0\}\rangle \to |\{n_j\}\rangle\,, \quad \sum_j n_j = kN,$$

$$\langle\{0\}|\Big(\prod_{s=1}^{i} \mathbf{E}^{-}_{\ell_s,P}\Big) \to \langle\{n'_j\}|\,, \quad \sum_j n'_j = iN,$$

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we find terms in the expansion are non-vanishing if and only if the number of creation and annihilation operators is equal.

$$\langle\mathcal{Y}^{Q}_{\mathbf{0}}|\mathcal{Y}^{P}_{\mathbf{0}}\rangle = \prod_{j=1}^{m'} s'^{j}_{11}\prod_{j=1}^{m} s^{j}_{22}\Big[1+\sum_{j=1}^{m'}\sum_{\ell=1}^{m} y'_j y_\ell\langle\{0\}|\mathbf{E}^{-}_{j,Q}\mathbf{E}^{+}_{\ell,P}|\{0\}\rangle+\cdots$$

$$+\sum_{j_1\le\cdots\le j_n}\!\!\cdots\sum\sum_{\ell_1\le\cdots\le\ell_n}\!\!\cdots\sum(y_{\ell_1}\cdots y_{\ell_n})(y'_{j_1}\cdots y'_{j_n})\langle\{0\}|\prod_{s=1}^{n}\mathbf{E}^{-}_{j_s,Q}\prod_{s=1}^{n}\mathbf{E}^{+}_{\ell_s,P}|\{0\}\rangle$$

$$+\cdots+(y_1\cdots y_m)(y'_1\cdots y'_{m'})\langle\{0\}|\prod_{j=1}^{m'}\mathbf{E}^{-}_{j,Q}\prod_{\ell=1}^{m}\mathbf{E}^{+}_{\ell,P}|\{0\}\rangle.$$

Inner Product of the Maximum Eigenvectors

Let

$$W_n = (\ell_1, \cdots, \ell_n), \quad W'_n = (j_1, \cdots, j_n)$$

then we may rewrite the previous equation as

Go to DPQ

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$$\langle \mathcal{Y}_{\mathbf{0}}^{Q} | \mathcal{Y}_{\mathbf{0}}^{P} \rangle = \prod_{j=1}^{m'} s_{11}^{\prime j} \prod_{j=1}^{m} s_{22}^{j} \sum_{s} \sum_{s'} (y_1)^{s_1} (y_2)^{s_2} \cdots (y_m)^{s_m}$$

$$\times \langle \{0\} | \prod_{i \in W'_n} \mathbf{E}_{i,Q}^{-} \prod_{j \in W_n} \mathbf{E}_{j,P}^{+} | \{0\} \rangle (y'_1)^{s'_1} (y'_2)^{s'_2} \cdots (y'_{m'})^{s'_{m'}}.$$

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$$\times \langle \{0\}| \prod_{i \in W'_n} \mathbf{E}^{-}_{i,Q} \prod_{j \in W_n} \mathbf{E}^{+}_{j,P} |\{0\}\rangle (y'_1)^{s'_1} (y'_2)^{s'_2} \cdots (y'_{m'})^{s'_{m'}}\,.$$

Similarly for $P \leftrightarrow Q$ we have

$$\langle \mathcal{Y}_{\mathbf{0}}^{P} | \mathcal{Y}_{\mathbf{0}}^{Q} \rangle = \prod_{j=1}^{m} s_{11}^{j} \prod_{j=1}^{m'} s_{22}^{\prime j} \sum_{s} \sum_{s'} (\hat{y}_1)^{s_1} (\hat{y}_2)^{s_2} \cdots (\hat{y})_m^{s_m}$$
$$\times \langle \{0\}| \prod_{j \in W_n} \mathbf{E}^{-}_{j,P} \prod_{i \in W'_n} \mathbf{E}^{+}_{i,Q} |\{0\}\rangle (\hat{y}'_1)^{s'_1} (\hat{y}'_2)^{s'_2} \cdots (\hat{y}'_{m'})^{s'_{m'}}\,,$$

where

$$y_j = s_{12}^{j}/s_{22}^{j} \rightarrow \hat{y}'_j = s_{12}^{\prime j}/s_{22}^{\prime j}\,, \quad y'_j = -s_{21}^{\prime j}/s_{11}^{\prime j} \rightarrow \hat{y}_j = -s_{21}^{j}/s_{11}^{j}\,.$$

Homogeneous case with $p' = p$

Since $\mu_p = 1/\mu_{p'}$ we have $\mu_p = 1$. Setting $\lambda_p = \mu_p^N = 1$ in the equations for s_{ij} which are in Appendix C of paper II, we find

$$\hat{y}_j = -z_j y_j\,, \quad \hat{y}'_j = (-z'_j)^{-1} y'_j.$$

Relating to the Variables of Baxter's

Let $w_j = 1/z_j$ so that $c_j = -(1+z_j)/(1-z_j)$ and

$$\lambda_j = \sqrt{1 + k'^2 + 2k'(1+z_j)/(1-z_j)}$$

we find

$$y_j = \frac{\lambda_j - 1 + k'}{k' + 1 + \lambda_j} = -y_j'^B\,, \quad y'_j = -\frac{\lambda'_j - 1 - k'}{1 + \lambda'_j - k'} = -y_j^B.$$

and

$$s_{11}^{j,P} s_{22}^{j,P} = \frac{(\lambda_j + 1)^2 - k'^2}{4\lambda_j}$$

which is the inverse of Z_P in (Baxter.IV.3.38), as should be.

Proposition

Comparing with the sum of Baxter, ▸ Go to inner we need to have

$$\langle\{0\}|\prod_{i\in W_n'}\mathbf{E}^{-}_{i,Q}\prod_{j\in W_n}\mathbf{E}^{+}_{j,P}|\{0\}\rangle=\langle\{0\}|\prod_{j\in W_n}z_j\mathbf{E}^{-}_{j,P}\prod_{i\in W_n'}z_i'^{-1}\mathbf{E}^{+}_{i,Q}|\{0\}\rangle\propto\frac{A_{s,s'}B_{s,s'}}{C_sD_{s'}}.$$

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Proposition for $P \geq Q$

We shall prove by induction the following

$$\langle\{0\}|\prod_{i\in W'_n}\mathbf{E}^-_{i,Q}\prod_{j\in W_n}\mathbf{E}^+_{j,P}|\{0\}\rangle=\frac{\bar{A}_{s,s'}\bar{B}_{s,s'}}{\bar{C}_s\bar{D}_{s'}}$$

$$\bar{A}_{s,s'}=\prod_{i\in W_n}\prod_{j\in V'_n}(z_i-z'_j),\quad \bar{B}_{s,s'}=\prod_{i\in W'_n}\prod_{j\in V'_n}(z'_i-z_j),$$

$$\bar{C}_s=\prod_{i\in W_n}\prod_{j\in V_n}(z_i-z_i),\quad \bar{D}_{s'}=\prod_{i\in W'_n}\prod_{j\in V'_n}(z'_i-z'_j),$$

Proof of Proposition by Induction:

Step 1: $n = 1$

We shall first try to prove it for $n = 1$. Goto EE Goto ee1

$$\langle \{0\} | \mathbf{E}^{-}_{j,Q} \mathbf{E}^{+}_{\ell,P} | \{0\} \rangle = - \frac{\beta^{Q}_{j,0} \beta^{P}_{\ell,0} z_\ell}{\Lambda^{Q}_{0} \Lambda^{P}_{0}} \sum_{\substack{\{0 \le n_j \le N-1\} \\ n_1 + \cdots + n_L = N}} \bar{G}_Q(\{n_j\}, z'_j) G_P(\{n_j\}, z_\ell).$$

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Define the function

$$\mathrm{h}^{Q,P}(z'_k,z) \equiv \sum_{\substack{\{0\le n_j\le N-1\}\\ n_1+\cdots+n_L=N}}\bar{G}_Q(\{n_j\},z'_k)G_P(\{n_j\},z).$$

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Goto EE

$$\mathrm{h}^{Q,P}(z'_k, z) = \sum_{\ell=0}^{m'-1} \sum_{j=0}^{m-1} z'^{\ell}_k z^j \sum_{\substack{\{0 \le n_j \le N-1\} \\ n_1 + \cdots + n_L = N}} \bar{K}_{\ell N + Q}(\{n_j\}) K_{jN+P}(\{n_j\})$$

Conjecture on the Coefficients of $\mathcal{G}(t, u)$

Symmetry

▸ Goto Gtu

$$\mathrm{h}^{Q,P}(z'_k, z) = \sum_{\ell=0}^{m'-1} \sum_{j=0}^{m-1} z'^{\ell}_k \, z^j \mathcal{G}_{\ell N+Q, jN+P}$$

$$= \sum_{\ell=0}^{m'-1} \sum_{j=0}^{m-1} z'^{\ell}_k \, z^j \mathcal{G}_{jN+P, \ell N+Q} = \mathrm{h}^{P,Q}(z, z'_k)$$

due to $\mathcal{G}_{\ell,k} = \mathcal{G}_{k,\ell}$. ▸ Goto Gtu ▸ Goto Proofn1

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$$\mathrm{h}^{Q,P}(z'_k, z) = \sum_{\ell=0}^{m'-1} \sum_{j=0}^{m-1} z'^{\ell}_k z^j \mathcal{G}_{\ell N+Q, jN+P}$$

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due to $\mathcal{G}_{\ell,k} = \mathcal{G}_{k,\ell}$. Goto Gtu Goto Proofn1

Conjecture

If Λ^Q_ℓ and Λ^P_ℓ denote the coefficients of the polynomial $P_Q(z)$ and $P_P(z)$, then for $P \geq Q$ we have Goto pf(qq)

$$\mathcal{G}_{\ell N+Q, jN+P} = \mathcal{G}_{jN+P, \ell N+Q} =$$

$$= \sum_{n=0}^{j} [(j-n+1)\Lambda^Q_n \Lambda^P_{\ell+1+j-n} - (n-\ell)\Lambda^Q_{\ell+1+j-n}\Lambda^P_n].$$

Proof for $P = Q$ for the Coefficients of $\mathcal{G}(t, u)$

$$\mathcal{G}_{\ell N+Q, jN+P} \equiv \sum_{\substack{\{0 \le n_j \le N-1\} \\ \sum n_j = N}} \bar{K}_{\ell N+Q}(\{n_j\}) K_{jN+P}(\{n_j\})$$

$$= \sum_{\substack{\{0 \le \lambda_j, n_j, n'_j \le N-1\} \\ \sum n_j = N, \sum n'_j = \ell N+Q, \sum \lambda_j = jN+P}} \prod_{j=1}^{L} \begin{bmatrix} n'_j + n_j \\ n_j \end{bmatrix} \begin{bmatrix} n_j + \lambda_j \\ n_j \end{bmatrix} \omega^{n_j(N'_j + \bar{b}_j)} \quad \boxed{\bar{b}_j = \sum_{i>j} \lambda_i}$$

$$\boxed{\mu_j = n_j + n'_j}$$

Proof for $P = Q$ for the Coefficients of $\mathcal{G}(t, u)$

$$\mathcal{G}_{\ell N+Q, jN+P} \equiv \sum_{\substack{\{0\le n_j \le N-1\} \\ \sum n_j = N}} \bar{K}_{\ell N+Q}(\{n_j\}) K_{jN+P}(\{n_j\})$$

$$= \sum_{\substack{\{0\le \lambda_j, n_j, n'_j \le N-1\} \\ \sum n_j = N, \sum n'_j = \ell N+Q, \sum \lambda_j = jN+P}} \prod_{j=1}^{L} \begin{bmatrix} n'_j + n_j \\ n_j \end{bmatrix} \begin{bmatrix} n_j + \lambda_j \\ n_j \end{bmatrix} \omega^{n_j(N'_j + \bar{b}_j)} \qquad \boxed{\bar{b}_j = \sum_{i>j} \lambda_i}$$

$$\boxed{\mu_j = n_j + n'_j} = \sum_{\substack{\{0\le \lambda_j, n_j, n'_j \le N-1\} \\ \sum n_j = N, \sum \mu_j = \ell N+N+Q, \sum \lambda_j = jN+P}} \prod_{j=1}^{L} \begin{bmatrix} \mu_j \\ n_j \end{bmatrix} \begin{bmatrix} n_j + \lambda_j \\ n_j \end{bmatrix} \omega^{n_j(a_j - N_j + \bar{b}_j)}$$

Proof for $P = Q$ for the Coefficients of $\mathcal{G}(t,u)$

$$\mathcal{G}_{\ell N+Q, jN+P} \equiv \sum_{\substack{\{0\le n_j \le N-1\} \\ \sum n_j = N}} \bar{K}_{\ell N+Q}(\{n_j\}) K_{jN+P}(\{n_j\})$$

$$= \sum_{\substack{\{0\le \lambda_j, n_j, n'_j \le N-1\} \\ \sum n_j = N, \sum n'_j = \ell N+Q, \sum \lambda_j = jN+P}} \prod_{j=1}^{L} \begin{bmatrix} n'_j + n_j \\ n_j \end{bmatrix} \begin{bmatrix} n_j + \lambda_j \\ n_j \end{bmatrix} \omega^{n_j(N'_j + \bar{b}_j)} \quad \boxed{\bar{b}_j = \sum_{i>j} \lambda_i}$$

$$\boxed{\mu_j = n_j + n'_j} = \sum_{\substack{\{0\le \lambda_j, n_j, n'_j \le N-1\} \\ \sum n_j = N, \sum \mu_j = \ell N+N+Q, \sum \lambda_j = jN+P}} \prod_{j=1}^{L} \begin{bmatrix} \mu_j \\ n_j \end{bmatrix} \begin{bmatrix} n_j + \lambda_j \\ n_j \end{bmatrix} \omega^{n_j(a_j - N_j + \bar{b}_j)}$$

$$\boxed{a_j = \sum_{i<j} \lambda_i, \; N_j = \sum_{i<j} n_i} \quad = \sum_{\substack{\{0\le \mu_j \le N-1\} \\ \sum \mu_j = (\ell+1)N+Q}} \sum_{\substack{\{0\le \lambda_j \le N-1\} \\ \sum \lambda_j = jN+P}} \mathcal{I}_N(\{\mu_j\};\{\lambda_j\}).$$

Proof for $P = Q$ for the Coefficients of $\mathcal{G}(t,u)$

$$\mathcal{G}_{\ell N+Q, jN+P} \equiv \sum_{\substack{\{0\le n_j\le N-1\}\\ \sum n_j = N}} \bar{K}_{\ell N+Q}(\{n_j\}) K_{jN+P}(\{n_j\})$$

$$= \sum_{\substack{\{0\le \lambda_j, n_j, n'_j \le N-1\}\\ \sum n_j = N, \sum n'_j = \ell N+Q, \sum \lambda_j = jN+P}} \prod_{j=1}^{L} \begin{bmatrix} n'_j + n_j \\ n_j \end{bmatrix} \begin{bmatrix} n_j + \lambda_j \\ n_j \end{bmatrix} \omega^{n_j(N'_j + \bar{b}_j)}$$

$$\boxed{\bar{b}_j = \sum_{i>j} \lambda_i}$$

$$\boxed{\mu_j = n_j + n'_j}$$

$$= \sum_{\substack{\{0\le \lambda_j, n_j, n'_j \le N-1\}\\ \sum n_j = N, \sum \mu_j = \ell N+N+Q, \sum \lambda_j = jN+P}} \prod_{j=1}^{L} \begin{bmatrix} \mu_j \\ n_j \end{bmatrix} \begin{bmatrix} n_j + \lambda_j \\ n_j \end{bmatrix} \omega^{n_j(a_j - N_j + \bar{b}_j)}$$

$$\boxed{a_j = \sum_{i<j} \lambda_i, \; N_j = \sum_{i<j} n_i}$$

$$= \sum_{\substack{\{0\le \mu_j \le N-1\}\\ \sum \mu_j = (\ell+1)N+Q}} \sum_{\substack{\{0\le \lambda_j \le N-1\}\\ \sum \lambda_j = jN+P}} \mathcal{I}_N(\{\mu_j\};\{\lambda_j\}).$$

$$\mathcal{I}_m(\{\mu_j\};\{\lambda_j\}) \equiv \sum_{\substack{\{0\le n_j\le N-1\}\\ n_1+\cdots+n_L = m}} \prod_{j=1}^{L} \begin{bmatrix} \mu_j \\ n_j \end{bmatrix} \begin{bmatrix} n_j + \lambda_j \\ n_j \end{bmatrix} \omega^{n_j(a_j - N_j) + n_j \bar{b}_j}.$$

Identity for $P = Q$

For $a_{L+1} = \sum \mu_j = (\ell+1)N + Q$ and $\bar{b}_0 = \sum \lambda_j = jN + P$, we find from (III.28),(III.33) and (III.34) in paper III the relation

$$\sum_{n=0}^{a_{L+1}} (-1)^n \omega^{n^2} \mathcal{I}_n(\{\mu_j\};\{\lambda_j\}) t^n = (\omega^{\frac{1}{2}+P} t;\omega)_{N-P+Q} \times (1+t^N)^{\ell-j} \sum_{n=0}^{\bar{b}_0} (-1)^n \omega^{n^2} \bar{\mathcal{I}}_n(\{\lambda_j\};\{\mu_j\}) t^n,$$

where Goto difficulty

$$\bar{\mathcal{I}}_n(\{\lambda_j\};\{\mu_j\}) = \sum_{\substack{\{0\le n_j\le N-1\} \\ n_1+\cdots+n_L=n}} \prod_{j=1}^{L} \begin{bmatrix} \lambda_j \\ n_j \end{bmatrix} \begin{bmatrix} n_j+\mu_j \\ n_j \end{bmatrix} \omega^{n_j(\bar{b}_j-\bar{N}_j)+n_j a_j}.$$

By equating the coefficients of t^n on both sides of this equation, we relate the sums $\mathcal{I}_n$ and $\bar{\mathcal{I}}_n$. For $P = Q$, we find

$$\mathcal{I}_N(\{\mu_j\};\{\lambda_j\}) = (\ell - j + 1) + \bar{\mathcal{I}}_N(\{\lambda_j\};\{\mu_j\})$$

Proof of Conjecture for $P = Q$

$$\mathcal{G}_{\ell N+Q, jN+Q} = \sum_{\substack{\{0 \le \mu_i \le N-1\} \\ \sum \mu_i = (\ell+1)N+Q}} \sum_{\substack{\{0 \le \lambda_i \le N-1\} \\ \sum \lambda_i = jN+Q}} \mathcal{I}_N(\{\mu_i\}; \{\lambda_i\})$$

$$= \sum_{\substack{\{0 \le \mu_i \le N-1\} \\ \sum \mu_i = (\ell+1)N+Q}} \sum_{\substack{\{0 \le \lambda_i \le N-1\} \\ \sum \lambda_i = jN+Q}} [(\ell - j + 1) + \bar{\mathcal{I}}_N(\{\lambda_i\}; \{\mu_i\})]$$

$$= (\ell - j + 1)\Lambda^Q_{\ell+1}\Lambda^Q_j + \mathcal{G}_{\ell N+N+Q, jN-N+Q}$$

by letting $\lambda_i = \lambda'_i + n_i$.

Proof of Conjecture for $P = Q$

$$\mathcal{G}_{\ell N+Q, jN+Q} = \sum_{\substack{\{0\le\mu_i\le N-1\}\\ \sum\mu_i=(\ell+1)N+Q}} \sum_{\substack{\{0\le\lambda_i\le N-1\}\\ \sum\lambda_i=jN+Q}} \mathcal{I}_N(\{\mu_i\};\{\lambda_i\})$$

$$= \sum_{\substack{\{0\le\mu_i\le N-1\}\\ \sum\mu_i=(\ell+1)N+Q}} \sum_{\substack{\{0\le\lambda_i\le N-1\}\\ \sum\lambda_i=jN+Q}} [(\ell-j+1)+\bar{\mathcal{I}}_N(\{\lambda_i\};\{\mu_i\})]$$

$$= (\ell-j+1)\Lambda^Q_{\ell+1}\Lambda^Q_j + \mathcal{G}_{\ell N+N+Q, jN-N+Q}$$

by letting $\lambda_i = \lambda'_i + n_i$. For $j = 0$, $\bar{\mathcal{I}}_N = 0$, we have

$$\mathcal{G}_{\ell N+Q,Q} = (\ell+1)\Lambda^Q_{\ell+1}\Lambda^Q_0$$

Repeat the process until arriving at $\mathcal{G}_{\ell N+jN+Q,Q}$, which is given by the above equation. We find

$$\mathcal{G}_{\ell N+Q, jN+Q} = \sum_{n=0}^{j}(\ell+j+1-2n)\Lambda^Q_{\ell+1+j-n}\Lambda^Q_n.$$

Goto conjecture

Proof of Conjecture for $P = Q$

$$\mathcal{G}_{\ell N+Q, jN+Q} = \sum_{\substack{\{0\le\mu_i\le N-1\} \\ \sum\mu_i=(\ell+1)N+Q}} \sum_{\substack{\{0\le\lambda_i\le N-1\} \\ \sum\lambda_i=jN+Q}} \mathcal{I}_N(\{\mu_i\};\{\lambda_i\})$$

$$= \sum_{\substack{\{0\le\mu_i\le N-1\} \\ \sum\mu_i=(\ell+1)N+Q}} \sum_{\substack{\{0\le\lambda_i\le N-1\} \\ \sum\lambda_i=jN+Q}} [(\ell-j+1)+\bar{\mathcal{I}}_N(\{\lambda_i\};\{\mu_i\})]$$

$$= (\ell-j+1)\Lambda^Q_{\ell+1}\Lambda^Q_j + \mathcal{G}_{\ell N+N+Q, jN-N+Q}$$

by letting $\lambda_i = \lambda'_i + n_i$. For $j = 0$, $\bar{\mathcal{I}}_N = 0$, we have

$$\mathcal{G}_{\ell N+Q,Q} = (\ell+1)\Lambda^Q_{\ell+1}\Lambda^Q_0$$

Repeat the process until arriving at $\mathcal{G}_{\ell N+jN+Q,Q}$, which is given by the above equation. We find

$$\mathcal{G}_{\ell N+Q, jN+Q} = \sum_{n=0}^{j}(\ell+j+1-2n)\Lambda^Q_{\ell+1+j-n}\Lambda^Q_n.$$

Goto conjecture This proves the conjecture for $P = Q$.

Difficulties for $P > Q$

▸ Goto lbl For $P \neq Q$, we still can use that equation to relate $\mathcal{I}_n$ and $\bar{\mathcal{I}}_n$ and obtain

$$\mathcal{G}_{\ell N+Q,jN+P} = (\ell - j)\Lambda^Q_{\ell+1}\Lambda^P_j + \mathcal{G}_{\ell N+N+Q,jN-N+P} + \mathcal{U}^{Q,P}_{\ell,j}$$

where

$$\mathcal{U}^{Q,P}_{\ell,j} = \sum_{k=0}^{Q} \begin{bmatrix} N-P+Q \\ Q-k \end{bmatrix} \omega^{k^2-kP} \sum_{\{0\le\mu'_j\le N-1\}} \sum_{\{0\le\lambda'_j\le N-1\}} \mathcal{I}_{P-k}(\{\mu'_j\};\{\lambda'_j\})$$

$$\boxed{\sum \mu'_j = \ell N + N + Q + P - k,\ \sum \lambda'_j = jN + k}$$

is very complicated. We have checked for different P, Q and N the following result holds

$$\mathcal{U}^{Q,P}_{\ell,j} = \sum_{n=0}^{j} \Lambda^Q_n \Lambda^P_{\ell+1+j-n} - \sum_{n=0}^{j-1} \Lambda^P_n \Lambda^Q_{\ell+1+j-n}$$

Relation with the Cyclic Elementary Polynomials

If $f(a) = f(a+N)$, define

$$p_n = \sum_{a=1}^{N} [f(a)]^n\,, \quad h_n = \sum_{1\le a_1 \le a_2 \cdots \le a_n \le N} \sum \cdots \sum f(a_1) f(a_1) \cdots f(a_n)$$

then it is easy to prove

$$n h_n = \sum_{r=1}^{n} p_r h_{n-r}\,, \quad h_0 = 1\,, \quad h_1 = p_1\,, \quad 2h_2 = p_1^2 + p_2.$$

h_n can be obtained from p_n iteratively.

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$$\mathcal{G}(t,u) = \sum_{\substack{\{0\le n_j\le N-1\} \\ n_1+\cdots+n_L=N}} \frac{(1-t^N)^{L-1}(1-u^N)^{L-1}}{\prod_{j=1}^{L}(1-t\omega^{N_j})(1-u\omega^{-N_j})}, \; N_j = \sum_{i<j} n_i .$$

Since $N_1 = 0$, we may rewrite this equation as

$$\mathcal{G}(t,u) = \frac{1}{(1-t)(1-u)} \sum_{\{N_j\}} \prod_{j=2}^{L} f(N_j), \; f(a) = \frac{(1-t^N)(1-u^N)}{(1-t\omega^a)(1-u\omega^{-a})}$$

Iteration for the Generating Function $\mathcal{G}(t, u)$

Since $0 \leq n_j = N_{j+1} - N_j \leq N - 1$ so that

$$N_j \leq N_{j+1} \leq N_j + N - 1$$

Since $N_{L+1} = N$, we also have $0 \leq N_2 \leq N_3 \cdots \leq N_L \leq N_{L+1} = N$. If $n_1 = n_2 \cdots = n_j = 0$ and $1 \leq n_{j+1} \leq N - 1$ for $j = 1..L - 2$, then $1 \leq N_{j+1} \leq N_{j+2} \cdots \leq N_L \leq N$ excluding the case with $N_{j+1} = N_{j+2} \cdots = N_L = N$. Consequently we find

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$$\mathcal{G}(t, u) = \frac{1}{(1-t)(1-u)} \left\{ \sum_{j=0}^{L-2} [f(0)]^j h_{L-1-j} - (L-1)[f(0)]^{L-1} \right\}$$

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$$\begin{aligned} p_{n+1} &= \frac{[(1-t^N)(1-u^N)]^{n+1}}{(n!)^2} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \frac{\mathrm{d}^n}{\mathrm{d}u^n} \sum_{a=1}^{N} \frac{1}{(t-\omega^{-a})(u-\omega^a)} \\ &= \frac{[(1-t^N)(1-u^N)]^{n+1}}{(n!)^2} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \frac{\mathrm{d}^n}{\mathrm{d}u^n} \frac{N(1-u^N t^N)}{(1-ut)(1-t^N)(1-u^N)} \end{aligned}$$

Proof of Proposition for $n = 1$

We are fully convinced that the conjecture on the coefficients of $\mathcal{G}(t,u)$ is correct. As a consequence, we can prove all the rest.

Goto hPQ

$$\mathrm{h}^{Q,P}(z'_k, z) = \sum_{\ell=0}^{m'-1} \sum_{j=0}^{m-1} z_k'^{\ell} z^j \sum_{n=0}^{j} \Big[(j-n+1)\Lambda_n^Q \Lambda_{\ell+1+j-n}^P - (n-\ell)\Lambda_{\ell+1+j-n}^Q \Lambda_n^P\Big]$$

$\boxed{m' \geq m}$

$$= \sum_{\ell=0}^{m'-1} \sum_{n=0}^{m-1} \sum_{j=n}^{m-1} z_k'^{\ell} z^j \Big[(j-n+1)\Lambda_n^Q \Lambda_{\ell+1+j-n}^P - (n-\ell)\Lambda_{\ell+1+j-n}^Q \Lambda_n^P\Big]$$

$\boxed{j \to i = \ell + 1 + j - n}$

$$= \sum_{\ell=0}^{m'-1} \sum_{n=0}^{m-1} \sum_{i=\ell+1}^{m+\ell-n} \left(\frac{z'_k}{z}\right)^{\ell} z^{i+n-1} \Big[(i-\ell)\Lambda_n^Q \Lambda_i^P - (n-\ell)\Lambda_i^Q \Lambda_n^P\Big]$$

$$= \sum_{\ell=0}^{m'} \sum_{n=0}^{m'} \sum_{i=\ell+1}^{m+\ell-n} \left(\frac{z'_k}{z}\right)^{\ell} z^{i+n-1} \Big[(i-\ell)\Lambda_n^Q \Lambda_i^P - (n-\ell)\Lambda_i^Q \Lambda_n^P\Big]$$

Continue1

Split the interval of summation of i into three parts,

$$[\ell+1, m+\ell-n] \to [0, m'] - [0, \ell] - [m+\ell-n+1, m']$$

$$\mathrm{h}^{Q,P}(z_k', z) = \sum_{\ell=0}^{m'} \left(\frac{z_k'}{z}\right)^{\ell} [\alpha(z) - \beta(z) - \gamma(z)].$$

$m' = m, m+1$

$n \leftrightarrow i$

$$\alpha(z) = \sum_{n=0}^{m'} \sum_{i=0}^{m'} [(i-\ell)\Lambda_n^Q \Lambda_i^P - (n-\ell)\Lambda_i^Q \Lambda_n^P] z^{i+n-1} = 0$$

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$$\gamma(z) = \sum_{n=0}^{m'} \sum_{i=m+\ell-n+1}^{m'} [(i-\ell)\Lambda_n^Q \Lambda_i^P - (n-\ell)\Lambda_i^Q \Lambda_n^P] z^{i+n-1}$$

$$= \sum_{n=\ell+1}^{m'} \sum_{i=m+\ell-n+1}^{m'} [(i-\ell)\Lambda_n^Q \Lambda_i^P - (n-\ell)\Lambda_i^Q \Lambda_n^P] z^{i+n-1}$$

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$$= \sum_{i=\ell+1}^{m'} \sum_{n=m+\ell-i+1}^{m'} [(i-\ell)\Lambda_n^Q \Lambda_i^P - (i-\ell)\Lambda_n^Q \Lambda_i^P] z^{i+n-1} = 0.$$

Continue2

$$\mathrm{h}^{Q,P}(z'_k, z) = -\sum_{\ell=0}^{m'} \left(\frac{z'_k}{z}\right)^{\ell} \beta(z) = \sum_{n=0}^{m'} \sum_{i=0}^{m'} z^{n+i-1} \Bigg[(n\Lambda_i^Q \Lambda_n^P - i\Lambda_n^Q \Lambda_i^P) \sum_{\ell=i}^{m'} \left(\frac{z'_k}{z}\right)^{\ell} - (\Lambda_i^Q \Lambda_n^P - \Lambda_n^Q \Lambda_i^P) \sum_{\ell=i}^{m'} \ell \left(\frac{z'_k}{z}\right)^{\ell} \Bigg].$$

Using

$$\sum_{\ell=i}^{m} u^{\ell} = \frac{u^{m+1} - u^i}{u-1}, \qquad \sum_{\ell=i}^{m} \ell u^{\ell-1} = \frac{\mathrm{d}}{\mathrm{d}u} \sum_{\ell=i}^{m} u^{\ell},$$

$$\sum_{j=0}^{m} \Lambda_j^P z_k^j = P_P(z_k) = 0\,, \quad \sum_{j=0}^{m'} \Lambda_j^Q z_k'^j = P_Q(z'_k) = 0,$$

we find

$$\mathrm{h}^{Q,P}(z'_j, z_\ell) = \frac{z'_j P_Q(z_\ell) P_P(z'_j)}{(z'_j - z_\ell)^2} = \mathrm{h}^{P,Q}(z_\ell, z'_j).$$

Proposition Proven for $n = 1$

$$\langle \{0\} | \mathbf{E}^{-}_{j,Q} \mathbf{E}^{+}_{\ell,P} | \{0\} \rangle = -\frac{\beta^{Q}_{j,0}\beta^{P}_{\ell,0} z_\ell}{\Lambda^{Q}_{0}\Lambda^{P}_{0}} \mathrm{h}^{Q,P}(z'_j, z_\ell)$$
$$= -\frac{\beta^{Q}_{j,0}\beta^{P}_{\ell,0} z_\ell z'_j P_Q(z_\ell) P_P(z'_j)}{(z'_j - z_\ell)^2 \Lambda^{Q}_{0}\Lambda^{P}_{0}}$$

From (III.16) and (IV.63) we have

$$P_P(z) = \Lambda_m \prod_{k=1}^{m} (z - z_k), \quad \beta_{\ell,0} = -\frac{\Lambda_0}{\Lambda_m z_\ell} \prod_{k=1,k\neq\ell}^{m} \frac{1}{(z_\ell - z_k)},$$

$$P_Q(z) = \Lambda'_m \prod_{k=1}^{m'} (z - z'_k), \quad \beta'_{j,0} = -\frac{\Lambda'_0}{\Lambda'_{m'} z'_j} \prod_{k=1,k\neq j}^{m'} \frac{1}{(z'_j - z'_k)}$$

$$\langle \{0\} | \mathbf{E}^{-}_{j,Q} \mathbf{E}^{+}_{\ell,P} | \{0\} \rangle = \prod_{k\neq j,k=1}^{m'} \frac{(z_\ell - z'_k)}{(z'_j - z'_k)} \prod_{k\neq\ell,k=1}^{m} \frac{(z'_j - z_k)}{(z_\ell - z_k)}.$$

▸ Goto Proposition This is the proof of the Proposition with one integer in the sets $\boxed{W_1 = \{\ell\}, W'_1 = \{j\}}$.

Proposition for any n

$$\langle\{0\}|\mathbf{E}^{-}_{\ell,P}\mathbf{E}^{+}_{j,Q}|\{0\}\rangle = -(\beta^{Q}_{j,0}\beta^{P}_{\ell,0}z'_j/\Lambda^{Q}_0\Lambda^{P}_0)\mathrm{h}^{P,Q}(z_\ell, z'_j)$$

$$= -(\beta^{Q}_{j,0}\beta^{P}_{\ell,0}z'_j/\Lambda^{Q}_0\Lambda^{P}_0)\mathrm{h}^{Q,P}(z'_j, z_\ell) = (z'_j/z_\ell)\langle\{0\}|\mathbf{E}^{-}_{j,Q}\mathbf{E}^{+}_{\ell,P}|\{0\}\rangle$$

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Preliminaries

Denote

$$\psi_n(W_n, W'_n) = \langle\{0\}|\prod_{j\in W'_n}\mathbf{E}^{-}_{j,Q}\prod_{i\in W_n}\mathbf{E}^{+}_{i,P}|\{0\}\rangle$$

$$\bar{\psi}_n(W_n, W'_n) = \langle\{0\}|\prod_{i\in W_n}\mathbf{E}^{-}_{i,P}\prod_{j\in W'_n}\mathbf{E}^{+}_{j,Q}|\{0\}\rangle$$

We have shown that for $n = 1$, the following holds

$$\psi_n(W_n, W'_n) = \frac{\bar{A}_{s,s'}\bar{B}_{s,s'}}{\bar{C}_s\bar{D}_{s'}},$$

$$\bar{\psi}_n(W_n, W'_n) = \psi_n(W_n, W'_n)\prod_{i\in W_n}z_i^{-1}\prod_{j\in W'_n}z'_j$$

Proof of Proposition for $n+1$

Assume these equations hold for n, we shall prove that it also holds for $n+1$. Consider

$$\psi_{n+1}(W_{n+1}, W'_{n+1}) = \langle\{0\}|\Big(\prod_{j\in W'_n}\mathbf{E}^{-}_{j,Q}\Big)\mathbf{E}^{-}_{\ell,Q}\mathbf{E}^{+}_{k,P}\Big(\prod_{i\in W_n}\mathbf{E}^{+}_{i,P}\Big)|\{0\}\rangle,$$

where $W_{n+1} = \{W_n, k\}$ and $W'_{n+1} = \{W'_n, \ell\}$, so that $V_{n+1} = V_n/\{k\}$ and $V'_{n+1} = V'_n/\{\ell\}$

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where $W_{n+1} = \{W_n, k\}$ and $W'_{n+1} = \{W'_n, \ell\}$, so that $V_{n+1} = V_n/\{k\}$ and $V'_{n+1} = V'_n/\{\ell\}$

- Because $(\mathbf{E}^+_{j,P})^2 = 0$ and $(\mathbf{E}^-_{j,Q})^2 = 0$, we find that

$$\psi_{n+1}(W_{n+1}, W'_{n+1}) = 0 \quad \text{if} \quad \ell \in W'_n \quad \text{or} \quad k \in W_n$$

$(\equiv z'_\ell \to z'_j$ for $j \in W'_n$ or $z_k \to z_i$ for $i \in W_n)$.

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- $\psi_{n+1}(W_{n+1}, W'_{n+1}) = 0$ if $z'_\ell \to z_i$ for $i \in V_{n+1}$ ($\mathbf{E}^-_{\ell,Q} \to \mathbf{E}^-_{\ell,P}$, if $\ell \notin W_{n+1} \equiv \ell \in V_{n+1}$), or if $z_k = z'_j$ for $j \in V'_{n+1}$

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- $\psi_{n+1}(W_{n+1}, W'_{n+1}) = \psi_n(W_n, W'_n)$ if $z'_\ell \to z_k$

Proof Continued1

$$
\begin{aligned}
\psi_{n+1}(W_{n+1}, W'_{n+1}) &= \prod_{j\in W'_n} \frac{z'_\ell - z'_j}{z_k - z'_j} \prod_{j\in W_n} \frac{z_k - z_j}{z'_\ell - z_j} \prod_{j\in V_{n+1}} \frac{z'_\ell - z_j}{z_k - z_j} \\
&\qquad \prod_{j\in V'_{n+1}} \frac{z_k - z'_j}{z'_\ell - z'_j} \left[\frac{\phi(z_k)}{\phi(z'_\ell)}\right] \psi_n(W_n, W'_n) \\
&= \left[\prod_{j\in V'_{n+1}} (z_k - z'_j) \prod_{i\in W_n} \prod_{j\in V'_n} (z_i - z'_j) \Big/ \prod_{i\in W_n} (z_i - z'_\ell) \right] \\
&\quad \left[\prod_{j\in V_{n+1}} (z'_\ell - z_j) \prod_{i\in W'_n} \prod_{j\in V_n} (z'_i - z_j) \Big/ \prod_{i\in W'_n} (z'_i - z_k) \right] \\
&\quad \left[\prod_{i\in W'_n} (z'_i - z'_\ell) \Big/ \prod_{j\in V'_{n+1}} (z'_\ell - z'_j) \prod_{i\in W'_n} \prod_{j\in V'_n} (z'_i - z'_j) \right] \\
&\left[\prod_{i\in W'_n} (z_i - z_k) \Big/ \prod_{j\in V_{n+1}} (z_k - z_j) \prod_{i\in W_n} \prod_{j\in V_n} (z_i - z_j) \right] \left[\frac{\phi(z_k)}{\phi(z'_\ell)}\right]
\end{aligned}
$$

where $\phi(z)$ is a function to be determined.

Proof Continued2

Since the proposition holds for n, we have substituted it into the above equation. Now using $V'_n = \{V'_{n+1}, \ell\}$, $W_{n+1} = \{W_n, k\}$, and

$$\prod_{i\in W_n}\prod_{j\in V'_n}(z_i - z'_j)\Big/\prod_{i\in W_n}(z_i - z'_\ell) = \prod_{i\in W_n}\prod_{j\in V'_{n+1}}(z_i - z'_j),$$

$$\prod_{j\in V'_{n+1}}(z_k - z'_j)\prod_{i\in W_n}\prod_{j\in V'_{n+1}}(z_i - z'_j) = \prod_{i\in W_{n+1}}\prod_{j\in V'_{n+1}}(z_i - z'_j)$$

we find

$$\psi_{n+1}(W_{n+1}, W'_{n+1}) = \frac{\prod_{i\in W_{n+1}}\prod_{j\in V'_{n+1}}(z_i - z'_j)\prod_{i\in W'_{n+1}}\prod_{j\in V_{n+1}}(z'_i - z_j)}{\prod_{i\in W_{n+1}}\prod_{j\in V_{n+1}}(z_i - z_j)\prod_{i\in W'_{n+1}}\prod_{j\in V'_{n+1}}(z'_i - z'_j)}$$

Since ψ_{n+1} is completely symmetric in the $n+1$ variables in W_{n+1} and W'_{n+1}, we find $\phi(z) = 1$.

Proof for $\bar{\psi}$

The same procedure works for $\bar{\psi}_{n+1}(W_{n+1}, W'_{n+1})$, but now $\phi(z) = 1/z$. ▸ Goto psi

$$\langle \mathcal{Y}_{\mathbf{0}}^{Q} | \mathcal{Y}_{\mathbf{0}}^{P} \rangle = \prod_{j=1}^{m'} s_{11}^{\prime j} \prod_{j=1}^{m} s_{22}^{j} \, D_{QP} \,, \quad \langle \mathcal{Y}_{\mathbf{0}}^{P} | \mathcal{Y}_{\mathbf{0}}^{Q} \rangle = \prod_{j=1}^{m} s_{11}^{j} \prod_{j=1}^{m'} s_{22}^{\prime j} \, D_{PQ}$$

where

$$D_{QP} = \sum_{s} \sum_{s'} (y_1)^{s_1} (y_2)^{s_2} \cdots (y_m)^{s_m} \psi_n(W_n, W'_n) (y'_1)^{s'_1} (y'_2)^{s'_2} \cdots (y'_{m'})^{s'_{m'}} \,,$$

$$\begin{aligned} D_{PQ} &= \sum_{s} \sum_{s'} (\hat{y}_1)^{s_1} (\hat{y}_2)^{s_2} \cdots (\hat{y})_m^{s_m} \bar{\psi}_n(W_n, W'_n) (\hat{y}'_1)^{s'_1} (\hat{y}'_2)^{s'_2} \cdots (\hat{y}'_{m'})^{s'_{m'}} \\ &= \sum_{s} \sum_{s'} (\hat{y}_1/z_1)^{s_1} (\hat{y}_2/z_2)^{s_2} \cdots (\hat{y}/z_m)_m^{s_m} \\ &\quad \times \psi_n(W_n, W'_n) (z'_1 \hat{y}'_1)^{s'_1} (z'_2 \hat{y}'_2)^{s'_2} \cdots (z'_{m'} \hat{y}'_{m'})^{s'_{m'}} = D_{QP} \end{aligned}$$

▸ Goto hom

Identical to Baxter's Sum

Since $c = (z+1)/(z-1)$, we find

$$z_i - z_j' = \frac{-2(c_i - c_j')}{(c_i - 1)(c_j' - 1)}\,, \quad z_i' - z_j = \frac{-2(c_i' - c_j)}{(c_i' - 1)(c_j - 1)}$$

$$\bar{A}_{s,s'} = \frac{(-2)^{n(m'-n)} A_{s,s'}}{\prod_{i\in W}(c_i-1)^{m'-n}\prod_{j\in V'}(c_j'-1)^n},\ \bar{B}_{s,s'} = \frac{(2)^{n(m-n)} B_{s,s'}}{\prod_{i\in W'}(c_i'-1)^{m-n}\prod_{j\in V}(c_j-1)^n},$$

$$\bar{C}_s = \frac{(2)^{n(m-n)} C_s}{\prod_{i\in W}(c_i-1)^{m-n}\prod_{j\in V}(c_j-1)^n}\,, \quad \bar{D}_{s'} = \frac{(-2)^{n(m'-n)} D_{s'}}{\prod_{i\in W'}(c_i'-1)^{m'-n}\prod_{j\in V'}(c_j'-1)^n}.$$

$$\frac{\bar{A}_{s,s'}\bar{B}_{s,s'}}{\bar{C}_s\bar{D}_{s'}} = \frac{A_{s,s'}B_{s,s'}}{C_s D_{s'}}\,\frac{\prod_{i\in W'}(c_i'-1)^{m'-m}}{\prod_{i\in W}(c_i-1)^{m'-m}}$$

For $m = m'$, the factors cancel out; and for $m' = m+1$, it gives the factors such that the result agrees with Baxter's.