Spontaneous Magnetization in the Integrable Chiral Potts Model

Cracking the Determinant

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Outline

- It has been now proven that the order parameter can be written as a sum D_{PQ}. Baxter further conjectured in early 2009 that D_{PQ} is a determinant. In the most recent paper he proved it. I shall go through his proof here. In this talk, I will include the verbatim version of his paper or the notes he sent me. As his English is better than mine, to change the writing would make it less understandable.

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- I will talk about how he cracked the determinant. During our stay in Canberra, Jacques was able to express it as a Fredholm determinant, but he and Bazhanov have spent some time trying to solve it, but did not suceed. It is worthwhile to understand Baxter's method.
- Finally I will discuss many of the other open problems that remain in the chiral Potts model.

Definitions

Let $c_1, \ldots, c_m, y_1, \ldots, y_m$ and $c'_1, \ldots, c'_{m'}, y'_1, \ldots, y'_n$ be sets of variables, with both m, n be arbitrary positive integers, and the c_i, y_i, c'_i, y'_i to be arbitrary variables.

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$$\kappa_s \ = \ s_1 + \cdots + s_m \,, \ \ \kappa_{s'} \ = \ s'_1 + \cdots + s'_n \,.$$

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$$\kappa_s = s_1 + \cdots + s_m \,, \quad \kappa_{s'} = s'_1 + \cdots + s'_n \,.$$

For a given set s, let V be the set of integers i such that $s_i = 0$ and W the set such that $s_i = 1$. The set V has $m - \kappa_s$ integers, while W has κ_s. Define V', W' similarly for the set s', so V' has $n - \kappa_{s'}$ elements, while W' has $\kappa_{s'}$.

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$$A_{s,s'} = \prod_{i \in W} \prod_{j \in V'} (c_i - c'_j) \quad , \quad B_{s,s'} = \prod_{i \in V} \prod_{j \in W'} (c_i - c'_j) \,,$$

$$C_s \ = \ \prod_{i \in W} \prod_{j \in V} (c_j - c_i) \quad , \quad D_{s'} \ = \ \prod_{i \in V'} \prod_{j \in W'} (c'_j - c'_i) \,.$$

$$\mathcal{R}_{mn} \ = \ \sum_{s} \sum_{s'} y_1^{s_1} y_2^{s_2} \cdots y_m^{s_m} \left(\frac{A_{s,s'} B_{s,s'}}{C_s D_{s'}} \right) y_1'^{\,s'_1} y_2'^{\,s'_2} \cdots y_n'^{\,s'_n} \,, \quad \boxed{\kappa_s = \kappa_{s'}}.$$

Definitions(continue)

Now define $a_i, \dots, a_m, b_1, \dots b_m$ and $a'_i, \dots, a'_n, b'_1, \dots b'_n$ by

$$a_i = \prod_{j=1}^{n} (c_i - c'_j)\,, \quad a'_i = \prod_{j=1}^{m} (c'_i - c_j)\,,$$

$$b_i = \prod_{j=1, j\neq i}^{m} (c_i - c_j)\,, \quad b'_i = \prod_{j=1, j\neq i}^{n} (c'_i - c'_j)\,,$$

▸ Goto pf-d1 ▸ Goto pf-ndd ▸ Goto pf-ccd and let $\mathcal{B}$ be an m by n matrix, and $\mathcal{B}'$ an n by m matrix, with elements

$$\mathcal{B}_{ij} = \frac{a_i}{b_i(c_i - c'_j)}\,, \quad \mathcal{B}'_{ij} = \frac{a'_i}{b'_i(c'_i - c_j)}\,.$$

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$$\mathcal{B}_{ij} = \frac{a_i}{b_i(c_i - c'_j)}\,, \quad \mathcal{B}'_{ij} = \frac{a'_i}{b'_i(c'_i - c_j)}\,.$$

Also, define an m by m diagonal matrix Y and an n by n diagonal matrix Y' by

$$Y_{i,j} \; = \; y_i \delta_{ij}\,, \quad Y'_{i,j} \; = \; y'_i \delta_{ij}\,.$$

Then the determinant mentioned above is

$$\mathcal{D}_{mn} \; = \; D_{PQ} \; = \; \det[I_m + Y\mathcal{B}Y'\mathcal{B}']\,,$$

where I_m is the identity matrix of dimension m.

Proof that $\mathcal{R}_{mn} = \mathcal{D}_{mn}$

Both $\mathcal{R}_{mn}$ and $\mathcal{D}_{mn}$ are rational functions of $c_1, \ldots c_m$, $c'_1, \ldots c'_n$. They are symmetric, being unchanged by simultaneously permuting the c_i and the y_i, as well as by simultaneously permuting the c'_i and the y'_i. We find that they are identical, for all c_i, y_i, c'_i, y'_i. The proof proceeds by recurrence, in the following four steps.

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1. The case $m = 1$

- Calculation of $\mathcal{R}_{1n}$: If $m = 1$ then $s = \{s_1\}$ and either $s_1 = 0$ or $s_1 = 1$. If $s_1 = 0$, $W = \emptyset$, so is W'; so that we get $A_{s,s'} = B_{s,s'} = C_s = D_{s'} = 1$. Goto ABCD

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$$A_{s,s'} = \prod_{j=1, j\neq r}^{n} (c_1 - c'_j) , \quad D_{s'} = \prod_{j=1, j\neq r}^{n} (c'_r - c'_j) .$$

It follows that

$$\mathcal{R}_{1n} \;=\; 1 + \sum_{r=1}^{n} y_1 y'_r \prod_{j=1, j\neq r}^{n} \frac{c_1 - c'_j}{c'_r - c'_j} .$$

Proof continued 1

- Calculation of $\mathcal{D}_{1n}$: The RHS of (1.1) is a determinant of dimension 1, so

$$\begin{aligned}\mathcal{D}_{1n} &= 1 + Y_{1,1}\sum_{r=1}^{n}\mathcal{B}_{1,r}Y'_{r,r}\mathcal{B}'_{r,1}\\ &= 1 - \frac{a_1 y_1}{b_1}\sum_{r=1}^{n}\frac{a'_r y'_r}{b'_r(c_1 - c'_r)^2},\end{aligned}$$

Goto ab

$$\boxed{a_1 = \prod_{j=1}^{n}(c_1 - c'_j),\ b_1 = 1}$$

$$\boxed{a'_r = (c'_r - c_1),\ b'_r = \prod_{j=1, j\neq r}^{n}(c'_r - c'_j)}$$

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$\boxed{a'_r = (c'_r - c_1),\ b'_r = \prod_{j=1, j\neq r}^{n}(c'_r - c'_j)}$ We therefore obtain

$$\mathcal{D}_{1n} = 1 + \sum_{r=1}^{n} y_1 y'_r \prod_{j=1, j\neq r}^{n}\frac{c_1 - c'_j}{c'_r - c'_j}$$

Proof continued 2

2: Degree of the numerator polynomials

Consider $\mathcal{R}_{mn}$ and $\mathcal{D}_{mn}$ as functions of c_m. They are both rational functions. We show here that they are both of the form

$$\frac{\text{polynomial of degree } (n-1)}{b_m}$$

Proof continued 2

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$$\frac{\text{polynomial of degree } (n-1)}{b_m}$$

- Degree for $\mathcal{R}_{mn}$: First consider the sum $\mathcal{R}_{mn}$ as a function of c_m. Goto ABCD Each term is plainly a polynomial divided by b_m. To determine the degree of this polynomial, consider the behaviour of $\mathcal{R}_{mn}$ when $c_m \to \infty$. Then, writing c_m simply as c. If $m \in W$, then the numerator is proportional to $A_{s,s'}$. As the degree of the numerator is the number of elements of V', $A_{s,s'} \sim c^{n-\kappa(s)}$. In the denominator $C_s \sim c^{m-\kappa(s)}$. As $b_m \sim c^{m-1}$, $A_{s,s'}/C_s \sim c^{n-m} = c^{n-1}/c^{m-1}$. The degree of the numerator is therefore at most $n-1$.
  If $m \in V$, then the numerator is proportional to $B_{s,s'} \sim c^{m-\kappa(s)}$ so that $B_{s,s'}/C_s \sim c^{m-1}/c^{m-1}$. The degree of the numerator is therefore at most $n-1$.

Proof continued 3

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- Degree for $\mathcal{D}_{mn}$: Now consider the determinant $\mathcal{D}_{mn}$ as a function of c_m. Goto ab
  At first sight there appear to be poles at $c_m = c'_j$, coming from $\mathcal{B}_{mj}$. However, they are cancelled by the factor a_m. Similarly, the ones in the element $\mathcal{B}'_{jm}$ of the matrix $\mathcal{B}'$ are cancelled by the factor a'_j. So there are no poles at $c_m = c'_j$, for any j.
  There are poles at $c_m = c_i$ (for $1 \le i < m$) coming from the b_i, b_m factors in $\mathcal{B}_{ij}, \mathcal{B}_{mj}$, respectively, so there are simple poles in each of the rows i and m. This threatens to create a double pole in the determinant $\mathcal{D}_{mn}$. However, if $c_m = c_i$, the rows i and m of the matrix $(c_m - c_i)\mathcal{B}$ are equal and opposite. By replacing row i by the sum of the two rows (corresponding to pre-multiplying $\mathcal{B}$ by an elementary matrix), we can eliminate the poles in row i. Hence there is only a single pole at $c_m = c_i$. The determinant is therefore a polynomial in c_m, divided by b_m.

Degree for $\mathcal{D}_{mn}$ continued

- To determine the degree of this polynomial, consider the behaviour of $\mathcal{D}_{mn}$ when $c_m = c \to \infty$.

$$\mathcal{B}_{ij} \sim c^{-1} \text{ if } i < m\,, \;\; \mathcal{B}_{ij} \sim c^{n-m} \text{ if } i = m\,,$$

$$\mathcal{B}'_{ij} \sim c \text{ if } j < m\,, \;\; \mathcal{B}'_{i,j} \sim 1 \text{ if } j = m\,.$$

and hence the orders of the elements of the matrix product are given by

$$Y\,\mathcal{B}\,Y'\,\mathcal{B}' \sim \begin{pmatrix} 1 & 1 & \dots & 1 & c^{-1} \\ 1 & 1 & \dots & 1 & c^{-1} \\ .. & .. & \dots & .. & .. \\ 1 & 1 & \dots & 1 & c^{-1} \\ c^{n-m+1} & c^{n-m+1} & \dots & c^{n-m+1} & c^{n-m} \end{pmatrix}.$$

Since $n \geq m$, it follows that $\mathcal{D}_{mn}$ grows at most as $\mathcal{D}_{mn} \sim c^{n-m}$. The numerator polynomial is therefore of degree at most $n-1$.

Proof continued 4

3. The case $c_m = c'_n$

- The sum $\mathcal{R}_{mn}$: Consider the case when $c_m = c'_n$. Goto ABCD
  If $m \in W$ and $n \in V'$, then from $A_{s,s'} = 0$. Similarly, If $m \in V$ and $n \in W'$, then from $B_{s,s'} = 0$. So the summand is zero unless either $m \in V, n \in V'$, or $m \in W, n \in W'$.
  In the first instance, $s_m = s'_n = 0$ ($m \in V$ or $n \in V'$.) The AB/CD factor is the same as if we replace m, n by $m-1, n-1$, respectively, except for a factor

$$\prod_{i \in W} \frac{c_i - c'_n}{c_m - c_i} \prod_{j \in W'} \frac{c_m - c'_j}{c'_j - c'_n}.$$

  Since $c_m = c'_n$, the factors in the product cancel, except for a sign. There as many elements in W as in W', so the sign products also cancel , leaving unity. Thus this contribution is exactly that obtained by replacing m, n by $m-1, n-1$.

Proof continued 5

- The sum $\mathcal{R}_{mn}$ continued: In the second instance, $s_m = s'_n = 1$ (or $m \in W$ or $n \in W'$.). This time AB/CD has an extra factor
$$\prod_{j \in V'} \frac{c_m - c'_j}{c'_n - c'_j} \prod_{i \in V} \frac{c_i - c'_n}{c_i - c_m} \;=\; 1\,,$$
so this contribution to the sum is again that obtained by replacing m, n by $m-1, n-1$, except that now there is an extra factor $y_m y'_n$. Adding the two contributions, we see that
$$\mathcal{R}_{mn} \;=\; (1 + y_m y'_n)\mathcal{R}_{m-1,n-1}\,.$$

Goto pf-rec

Proof continued 6

- The determinant $\mathcal{D}_{mn}$: Now look at the determinant when $c_m = c'_n$. Goto ab Since a_m and a'_n both contain the factor $c_m - c'_n$, the mth row of $\mathcal{B}$ vanishes except for the element m, n, which is
$$\mathcal{B}_{m,n} = \prod_{j=1}^{n-1}(c_m - c'_j) \Bigg/ \prod_{j=1}^{m-1}(c_m - c_j) \ .$$
Similarly, the nth row of $\mathcal{B}'$ vanishes except for
$$\mathcal{B}'_{n,m} = \prod_{j=1}^{m-1}(c'_n - c_j) \Bigg/ \prod_{j=1}^{n-1}(c'_n - c'_j) \ .$$
Since $c_m = c'_n$, we see that $\mathcal{B}_{m,n}\mathcal{B}'_{n,m} = 1$. It follows that the matrix is block-triangular

$\frac{a_i}{b_i} = \frac{\prod_{j=1}^{n-1}(c_i - c'_j)}{\prod_{j=1, j\neq i}^{m-1}(c_i - c_j)}$ same for a'_i/b'_i

$$I_m + Y\mathcal{B}Y'\mathcal{B}' = \begin{pmatrix} \mathbf{1} + \mathbf{y}\mathbf{b}\mathbf{y}'\mathbf{b}' & \cdots \\ \mathbf{0} & 1 + y_m y'_n \end{pmatrix} ,$$

where $\mathbf{1}, \mathbf{y}, \mathbf{b}, \mathbf{y}', \mathbf{b}'$ are the matrices for $\mathcal{D}_{m-1,n-1}$. Hence
$$\mathcal{D}_{m,n} = (1 + y_m y'_n)\mathcal{D}_{m-1,n-1} .$$

Proof continued 6

4. Proof by recurrence

The proof now proceeds by recurrence. Suppose $\mathcal{D}(m-1, n-1) = \mathcal{R}(m-1, n-1)$ for all c_i, c'_i. Then from the previous equations Goto pf-ccr1 it is true that $\mathcal{D}_{m,n} = \mathcal{R}_{m,n}$ when $c_m = c'_n$. By symmetry it is also true for $c_m = c'_j$ for $j = 1, \ldots, n$. Thus $\mathcal{D}_{m,n} - \mathcal{R}_{m,n}$ is zero for all these n values. But from point 2 above, this difference (times b_m) is a polynomial in c_m of degree $n-1$. The polynomial must therefore vanish identically, so $\mathcal{R}_{m,n} = \mathcal{D}_{m,n}$ for *arbitrary* values of c_m.
Since it is true for $m = 1$, it follows that

$$\mathcal{R}_{m,n} = \mathcal{D}_{m,n}$$

for *all* m, n.
This proves that the determinant is equal to the sum. Next, I will talk about the calculation of the determinant $\mathcal{D}_{PQ}$.

The expression for $\mathcal{D}_{PQ}$

$$D_{PQ} = \det[I_m + YBY'B^T]$$

where Y and Y' are diagonal matrices,

$$(Y)_{i,j} = y_i \delta_{ij}\,, \;\; (Y')_{i,j} = y'_i \delta_{ij},$$

while B is $m \times m'$ (notation change $n \to m'$) Cauchy-like matrix with elements

$$B_{ij} = \frac{f_i\, f'_j}{c_i - c'_j}\,.$$

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$$B_{ij} \;=\; \frac{f_i\, f'_j}{c_i - c'_j}\,.$$

The constants f_i and f_j are chosen as

$$f_i^2 \;=\; \frac{\epsilon a_i}{b_i}\,, \;\; f_i'^2 \;=\; -\frac{\epsilon\, a'_i}{b'_i}\,, \;\; \epsilon = \pm 1$$

so that B is orthogonal in the sense that

$$B^T B = I_{m'} \text{ if } m \geq m'\,, \;\; BB^T = I_m \text{ if } m \leq m'\,.$$

If F, F' are the diagonal matrices with elements $F_{i,i} = f_i$, $F'_{i,i} = f'_i$, then $\mathcal{B} = \epsilon F B F'^{-1}$, $\mathcal{B}' = -\epsilon F' B F^{-1}$ and $\epsilon^2 = 1$.

Alternative form for $\mathcal{D}_{PQ}$

Because B is orthogonal, we can write I_m in (1.1) as BB^T so that

$$\mathcal{D}_{PQ}(\alpha, \beta) \;=\; \det[U\, B^T]\,,$$

where

$$U \;=\; B + YBY'\,.$$

Thus the elements of U are

$$U_{ij} \;=\; \frac{f_i\, f_j'(1 + y_i y_j')}{c_i - c_j'}\,.$$

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Definition of Δ

$$\Delta_{m,m'}(c, c') \;=\; \frac{\displaystyle\prod_{1\le i<j\le m}(c_i - c_j)\prod_{1\le i<j\le m'}(c_j' - c_i')}{\displaystyle\prod_{i=1}^{m}\prod_{j=1}^{m'}(c_i - c_j')}\,.$$

The case $P > Q$, $m = m'$

The simplest case is when $m = m'$ and all matrices are square.

$$\det(B^T)\det(B) = 1 \Longrightarrow \mathcal{D}_{PQ} = \det U / \det B\,.$$

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Cauchy-like matrices

If A is the m by m matrix with entries

$$A_{ij} \;=\; \frac{1}{c_i - c'_j}\,,$$

then it is a Cauchy matrix and its determinant is $\Delta_{m,m}(c, c')$.
The matrix with elements

$$(B)_{ij} \;=\; \frac{-2\,k'\,f_i f'_j}{\lambda_i^2 - {\lambda'_j}^2}$$

is said to be Cauchy-like, and has determinant

$$\det B \;=\; \Delta_{m,m}(\lambda^2, {\lambda'}^2)\prod_{i=1}^{m}(-2k'\,f_i f'_i)\,.$$

The determinant of U

For $P > Q$, $m = m'$ we have

$$y_i \; = \; -\frac{1 + k' - \lambda_i}{1 - k' + \lambda_i} \, , \;\; y'_j \; = \; \frac{1 - k' - \lambda'_j}{1 + k' + \lambda'_j} \; .$$

Thus

$$1 + y_i y'_j \; = \; \frac{2\,(\lambda_i + \lambda'_j)}{(1 - k' + \lambda_i)(1 + k' + \lambda'_j)} \, .$$

Also

$$c_i - c'_j \; = \; ({\lambda'_j}^2 - {\lambda_i}^2)/2k' \, .$$

The determinant of U

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Thus

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Also

$$c_i - c'_j \;=\; ({\lambda'_j}^2 - {\lambda_i}^2)/2k'\,.$$

The elements of U are

$$U_{ij} \;=\; \frac{f_i\, f'_j(1+y_i y'_j)}{c_i - c'_j} \;=\; \frac{-\,4\,k'\,f_i f'_j}{(1-k'+\lambda_i)(1+k'+\lambda'_j)(\lambda_i-\lambda'_j)}$$

so U is a Cauchy-like matrix, similar to B. Analogously, its determinant is

$$\det U \;=\; \Delta_{m,m}(\lambda,\lambda')\;\prod_{i=1}^{m}\frac{-4k' f_i f'_i}{(1-k'+\lambda_i)(1+k'+\lambda'_i)}\,.$$

Order Parameter for $m = m'$

The f_i, f_i' cancel out of the ratio, leaving

$$\mathcal{D}_{PQ} = \frac{\Delta_{m,m}(\lambda, \lambda')}{\Delta_{m,m}(\lambda^2, \lambda'^2)} \prod_{i=1}^{m} \frac{2}{(1 - k' + \lambda_i)(1 + k' + \lambda_i')} .$$

We have

$$\left[\frac{\Delta_{m,m'}(\lambda, \lambda')}{\Delta_{m,m'}(\lambda^2, \lambda'^2)}\right]^2 = \frac{\prod_{i=1}^{m}\prod_{j=1}^{m'}(\lambda_i + \lambda_j')^2}{\prod_{1\le i<j\le m}(\lambda_i+\lambda_j)^2 \prod_{1\le i<j\le m'}(\lambda_j'+\lambda_i')^2} = \frac{\prod_{j=1}^{m'} 2\lambda_j'\mathcal{R}(\lambda_j')}{2^{(m-m')^2}\prod_{i=1}^{m}\mathcal{R}(\lambda_i)/2\lambda_i} .$$

where

$$\mathcal{R}(\lambda) = \frac{\prod_{i=1}^{m}(\lambda + \lambda_i)/2}{\prod_{j=1}^{m'}(\lambda + \lambda_j')/2} .$$

Order Parameter for $m = m'$ continued

Using

$$\mathcal{Z}_P = \prod_{i=1}^{m} \frac{4\lambda_i}{(1+\lambda_i)^2 - k'^2}, \quad \mathcal{Z}_Q = \prod_{i=1}^{m'} \frac{4\lambda'_i}{(1+\lambda'_i)^2 - k'^2}$$

and

$$[\mathcal{M}_r]^2 = \frac{D_{PQ}^2}{\mathcal{Z}_P \mathcal{Z}_Q},$$

we find

$$\left(\mathcal{M}_r^{(2)}\right)^2 = \frac{\mathcal{R}(1+k') \prod_{j=1}^{m'} \mathcal{R}(\lambda'_j)}{\mathcal{R}(1-k') \prod_{i=1}^{m} \mathcal{R}(\lambda_i)}.$$

Order Parameter for $m' = m + 1$

Now consider the case when $P > Q$ and $m' = m + 1$, so the matrices B, U are not square. We have

$$y_i = \frac{-2k'}{(1+\lambda_i)^2 - k'^2} \,, \quad y'_j = \frac{(1-\lambda'_j)^2 - k'^2}{2k'} \,.$$

Hence

$$1 + y_i y'_j \;=\; \frac{(\lambda_i + \lambda'_j)(2 + \lambda_i - \lambda'_j)}{(1+\lambda_i)^2 - k'^2} \,.$$

We see that the factor $\lambda_i + \lambda'_j$ again cancels, leaving

$$U \;=\; U^{(1)} + U^{(2)} \,,$$

where $U^{(1)}, U^{(2)}$ are matrices with elements

$$U^{(1)}_{ij} \;=\; \frac{-4k' f_i f'_j}{(\lambda_i - \lambda'_j)[(1+\lambda_i)^2 - k'^2)]} \,, \quad U^{(2)}_{ij} \;=\; \frac{-2k' f_i f'_j}{[(1+\lambda_i)^2 - k'^2]} \,,$$

respectively. Thus $U^{(2)} \;=\; \xi\eta^T$ where ξ, η are vectors and $(\eta)_j = f'_j$.

Consider the vector $B\eta$: it has entries $(B\eta)_i = f_i \mathcal{F}'(c_i)$ where

$$\mathcal{F}'(c) = \sum_{j=1}^{m'} \frac{f_j'^2}{c - c_j'} = \sum_{j=1}^{m'} \frac{\mathcal{P}_P(c_j')}{b_j'(c - c_j')} \quad \boxed{b_j' = \prod_{i\neq j, i=1}^{m'} (c_j' - c_i')}$$
$$= \mathcal{F}'(c) = \gamma' + \mathcal{P}_P(c)/\mathcal{P}_Q(c)\,,$$

where γ' is independent of c. Taking the limit $c \to \infty$, we obtain $\gamma' = 0$. It follows that $\mathcal{F}'(c)$ vanishes when $c = c_i$ $(i = 1, \ldots, m)$, so $B\,\eta = 0$. Consequently, the $U^{(2)}$ term is zero, leaving

$$\mathcal{D}_{PQ} = \det[U^{(1)} B^T]\,.$$

We define an m' by m' matrix

$$\hat{B} = \begin{pmatrix} B \\ \eta^T, \end{pmatrix}, \quad \boxed{c\mathcal{F}'(c) = \sum_{j=1}^{m'} \frac{c f_j'^2}{c - c_j'} = \frac{c\mathcal{P}_P(c)}{\mathcal{P}_Q(c)}}$$

$$\eta^T \eta = \sum_{j=1}^{m'} f_j'^2 = 1\,,$$

it follows that $\hat{B}\hat{B}^T = I_{m'}$, i.e. $\hat{B}$ is a square orthogonal matrix.

We also extend $U^{(1)}$ by adding the row η^T to form the square matrix

$$\mathcal{U} = \begin{pmatrix} U^{(1)} \\ \eta^T \end{pmatrix} \Rightarrow \mathcal{U}\hat{B}^T = \begin{pmatrix} U^{(1)}B^T & U^{(1)}\eta \\ \eta^T B^T & \eta^T\eta \end{pmatrix} = \begin{pmatrix} U^{(1)}B^T & U^{(1)}\eta \\ \mathbf{0} & 1 \end{pmatrix}$$

We see that $\mathcal{U}\hat{B}^T$ is an upper-right block triangular matrix and $\det\mathcal{U}\,\hat{B}^T = \det U^{(1)}B^T$. Using the orthogonality of $\hat{B}$,

$$\mathcal{D}_{PQ} \;=\; \det\mathcal{U}/\det\hat{B}\,.$$

The square matrices $\hat{B}$ and $\mathcal{U}$ are Cauchy-like, provided we take $f_{m+1} = -\lambda_{m+1}^2/2k'$ and then let $\lambda_{m+1} \to \infty$, so that $\hat{B}_{m',j} = -2k' f_{m+1} f'_j/(\lambda_{m+1}^2 - \lambda'^2_j) \to f'_j$, we obtain

$$\det\hat{B} \;=\; \Delta_{m,m'}(\lambda^2, \lambda'^2)\prod_{i=1}^{m}(2k' f_i)\prod_{j=1}^{m'} f'_j .$$

Similarly, all the elements $\mathcal{U}_{ij}$ of $\mathcal{U}$ are Cauchy like if now we take $f_{m+1} = -\lambda_{m+1}^3/4k'$ before letting $\lambda_{m+1} \to \infty$, so that

$$\mathcal{U}_{m+1,j} = -4k' f_{m+1} f_j' \Big/ (\lambda_{m+1} - \lambda_j)[(\lambda_{m+1} + 1)^2 - k'^2] \to f_j'$$

Again using the general formula, we find that

$$\det \mathcal{U} \;=\; \Delta_{m,m'}(\lambda, \lambda') \prod_{i=1}^{m} \frac{4k' f_i}{(1+\lambda_i)^2 - k'^2} \prod_{j=1}^{m'} f_j' \,.$$

Hence

$$\mathcal{D}_{PQ} \;=\; \frac{\Delta_{m,m'}(\lambda, \lambda')}{\Delta_{m,m'}(\lambda^2, \lambda'^2)} \prod_{i=1}^{m} \frac{2}{(1+\lambda_i)^2 - k'^2} \,.$$

We now obtain

$$\left(\mathcal{M}_r^{(2)}\right)^2 \;=\; \frac{\prod_{j=1}^{m'} \mathcal{R}(\lambda_j')}{\mathcal{R}(1+k')\,\mathcal{R}(1-k')\,\prod_{i=1}^{m} \mathcal{R}(\lambda_i)} \,.$$

Wiener-Hopf factorization

Let c, λ be two variables related as are c_i, λ_i. Noting that $\lambda_i^2 - \lambda^2 = 2k'(c - c_i)$, it follows from the definition of $\mathcal{R}$ that

$$\mathcal{R}(\lambda)\,\mathcal{R}(-\lambda) \;=\; (k'/2)^{m-m'}\mathcal{P}_P(c)/\mathcal{P}_Q(c),$$

where

$$\mathcal{P}_P(c) \;=\; \prod_{j=1}^{m}(c - c_j) = (c+1)^m \; P_P(w),$$

in which

$$P_P(t^N) \;=\; t^{-P}\sum_{n=0}^{N-1}\omega^{-nP}\left(\frac{1-t^N}{1-\omega^n t}\right)^L$$

$$= t^{-P}\left[\frac{1-t^N}{1-t} + \sum_{n=1}^{N-1}\omega^{-nP}\left(\frac{1-t^N}{1-\omega^n t}\right)^L\right]$$

Using these equations, this can be written as

$$\mathcal{R}(\lambda)\,\mathcal{R}(-\lambda) \;=\; [k'(c+1)/2]^{m-m'}\, t^{Q-P}\,\mathcal{W}(\lambda)\,,$$

Wiener-Hopf factorization1

where

$$\mathcal{W}(\lambda) = \mathcal{W}_P(\lambda)/\mathcal{W}_Q(\lambda),$$

$$\mathcal{W}_P(\lambda) = 1 + \sum_{n=1}^{N-1} \omega^{n(P+L)} \left(\frac{1-t}{\omega^n - t}\right)^L .$$

These t, w, c are related to λ. In particular,

$$t^N = w = \frac{c-1}{c+1} = \frac{(1-k')^2 - \lambda^2}{(1+k')^2 - \lambda^2} = \left[\frac{1-k'+\lambda}{1+k'+\lambda}\right]\left[\frac{1-k'-\lambda}{1+k'-\lambda}\right].$$

The function $\mathcal{R}(\lambda)$ is analytic in the RHS and is proportional to $(\lambda/2)^{m-m'}$ when $\lambda \to \infty$. It follows from

$$\frac{k'}{2}(c+1) = \frac{(1+k')^2 - \lambda^2}{4} = \left[\frac{1+k'+\lambda}{2}\right]\left[\frac{1+k'-\lambda}{2}\right],$$

that

$$\mathcal{R}(\lambda) = \left(\frac{1+\lambda+k'}{2}\right)^{m-m'} \left(\frac{1-k'+\lambda}{1+k'+\lambda}\right)^{(Q-P)/N} \mathcal{W}_+(\lambda).$$

where Wiener-Hopf factorization is performed on $\mathcal{W}(\lambda)$:

$$\mathcal{W}(\lambda) \;=\; \mathcal{W}_+(\lambda)\mathcal{W}_-(\lambda)\,,$$

and for $\Re(\lambda) \geq 0$,

$$\log \mathcal{W}_+(\lambda) \;=\; -\,\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\,\infty-\epsilon}^{\mathrm{i}\,\infty-\epsilon} \frac{\log \mathcal{W}(\lambda)}{\lambda' - \lambda}\, \mathrm{d}\lambda'$$

the integration being up a vertical line just to the left of the imaginary axis. Baxter has also shown that in the limit $L \to \infty$, $\mathcal{W}_+(\lambda) \to 1$ for for $\Re(\lambda) \geq 0$. It follows that

$$\frac{\prod_{j=1}^{m'} \mathcal{R}(\lambda'_j)}{\prod_{i=1}^{m} \mathcal{R}(\lambda_i)} \;=\; \mathcal{R}(1+k')^{m'-m} \left[\frac{\mathcal{R}(1+k')}{\mathcal{R}(1-k')}\right]^{(Q-P)/N}\,,$$

$$\mathcal{R}(1{+}k') \;=\; (1{+}k')^{m-m'+(P-Q)/N}\,,\;\; \mathcal{R}(1{-}k') \;=\; (1{-}k')^{(Q-P)/N}\,.$$

Order Parameter

m=m'

$$\left(\mathcal{M}_r^{(2)}\right)^2 = \frac{\mathcal{R}(1+k')\prod_{j=1}^{m}\mathcal{R}(\lambda_j')}{\mathcal{R}(1-k')\prod_{i=1}^{m}\mathcal{R}(\lambda_i)} = (1-k'^2)^{(P-Q)[N-(P-Q)]/N^2}$$
$$= (1-k'^2)^{r(N-r)/N^2}\,.$$

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m'=m+1

$$\left(\mathcal{M}_r^{(2)}\right)^2 = \frac{\prod_{j=1}^{m'}\mathcal{R}(\lambda'_j)/\prod_{i=1}^{m}\mathcal{R}(\lambda_i)}{\mathcal{R}(1+k')\,\mathcal{R}(1-k')}$$
$$= \frac{1}{\mathcal{R}(1-k')}\left[\frac{\mathcal{R}(1+k')}{\mathcal{R}(1-k')}\right]^{(Q-P)/N}$$
$$= (1-k'^2)^{r(N-r)/N^2}.$$

Open Questions

- Onsager Algebra

$$\sum_{k=-n}^{n} \alpha_{\pm k} A_{k-l} = 0, \quad \sum_{k=-n}^{n} \alpha_{\pm k} G_{k-l} = 0, \quad f(z) \equiv \sum_{k=-n}^{n} \alpha_k z^{k+n}.$$

What are these coefficients α_k? What is the dimension n?

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- Serre Relation not yet proven for all cases: The identity

$$\mathcal{I}_N(\{\mu_j\};\{\lambda_j\}) = (\ell - j + 1) + \bar{\mathcal{I}}_N(\{\lambda_j\};\{\mu_j\})$$

enables us to prove

$$[[[\mathbf{x}^{+}_{0,Q}, \mathbf{x}^{-}_{1,Q}], \mathbf{x}^{-}_{1,Q}]\mathbf{x}^{-}_{1,Q}](\mathbf{x}^{-}_{1,Q})^n|\{0\}\rangle = 0$$

Is there another identity which will enable us to prove the Serre relation?

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- Eigenvectors in

$$g_{2\ell}(r;k;q,\cdots,q) = \frac{1}{N}\sum_{Q=0}^{N-1}\sum_{j=1}^{J}\left[\frac{\Delta_j^P}{\Delta_{\max}^0}\right]^{2\ell}\langle\mathcal{Y}_{\max}^Q|\mathcal{Y}_j^P\rangle\langle\mathcal{Y}_j^P|\mathcal{Y}_{\max}^Q\rangle$$

  Do we need to know all the eigenvectors $|\mathcal{Y}_j^Q\rangle$?