PHYS 1114, Lecture 14, February 17

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1° Force example: Man pulling box.

2° Movie 1: Circular Motion: Direction of Centripetal Force
   a. Candles are placed on a platform and lit. Next glass chimneys are put around each of the candles. When the platform is spun the flames bend inwards showing the existence of centripetal forces on the flames.
   b. Because of Newton’s First Law the air wants to go straight. The colder air is denser ("heavier") than the flame and wins. The flame gets pushed inward by the cold air, which in turn is pushed inward by the chimney.

3. Movie 2: Circular Motion: Centrifugal Effects on a Rotating Sphere
   a. A set of brass hoops is mounted on a central axis as to model a sphere. At the top the hoops can slide freely over the axis. When the system is rotating the hoops flatten taking an elliptical shape. This is not due to outward forces. The necessary inward centripetal forces are provided by the bending and distortion of the hoops. These forces counter the effect of each hoop’s own inertia, the so-called centrifugal effect.
   b. This simple model explains why the Earth is flattened at the North and South Pole and elongated at the equator.

4° Uniform circular motion: Period, Tangential Velocity, and Centripetal Acceleration.
Example: Man pulling box

A man pulls a wooden box of mass $m = 50.0\,\text{kg}$ on a wooden floor by a rope making angle $\theta = 10.0^\circ$ with the horizontal.

Use: $\mu_s = 0.500$, $\mu_k = 0.200$.

1. Determine the minimum tension required to start the box moving.
2. What is the acceleration at that tension.

From Newton’s Second Law:

\[
\begin{align*}
F_{x,\text{net}} &= T \cos \theta - F_{\text{fr}} = ma_x, \\
F_{y,\text{net}} &= F_N + T \sin \theta - mg = 0.
\end{align*}
\]

1. If $F_{\text{fr}} = F_{\text{fr},s} < \mu_s F_N$, the box can not start moving, so that $v_x = a_x = 0$. Motion can start when $F_{\text{fr}} = F_{\text{fr},s} = \mu_s F_N$ is reached:

\[
\begin{align*}
T \cos \theta - \mu_s F_N &= 0, \\
F_N + T \sin \theta - mg &= 0.
\end{align*}
\]

\[\implies F_N = mg - T \sin \theta \quad \text{so that} \quad T \cos \theta - \mu_s(mg - T \sin \theta) = 0\]

\[\implies T(\cos \theta + \mu_s \sin \theta) = \mu_s mg \quad \implies \]

\[T = \frac{\mu_s mg}{\cos \theta + \mu_s \sin \theta} = \frac{0.500 \times 50.0\,\text{kg} \times 9.80\,\text{m/s}^2}{\cos 10.0^\circ + 0.500 \sin 10.0^\circ}
\]

\[= \frac{245.0\,\text{N}}{1.0716} = 228.6\,\text{N} = \boxed{2.29 \times 10^2\,\text{N}}\]
The values $\mu_s = 0.500$ and $\mu_k = 0.200$ are reasonable for wood on wood.

2. To find the original acceleration once the box moves:

$$\begin{cases} T \cos \theta - \mu_k F_N = ma_x, \\ F_N + T \sin \theta - mg = 0. \end{cases}$$

These are two equations with two unknowns, $a_x$ and $F_N$. Eliminating $F_N$, putting result from second equation into the first,

$$ma_x = T \cos \theta - \mu_k (mg - T \sin \theta)$$

$$= T(\cos \theta + \mu_k \sin \theta) - \mu_k mg$$

or

$$a_x = \frac{T(\cos \theta + \mu_k \sin \theta)}{m} - \mu_k g$$

$$= \frac{228.6 \text{ N} \times (\cos 10.0^\circ + 0.200 \sin 10.0^\circ)}{50.0 \text{ kg}} - 0.200 \times 9.80 \frac{\text{m}}{\text{s}^2}$$

$$= \frac{228.6 \text{ N} \times 1.0195}{50.0 \text{ kg}} - 1.960 \frac{\text{m}}{\text{s}^2}$$

$$= (4.661 - 1.960) \frac{\text{m}}{\text{s}^2}$$

$\implies a_x = 2.70 \frac{\text{m}}{\text{s}^2}$

(Note the different fonts used for $m = \text{mass}$ and $m = \text{meter}$.)
**Uniform Circular Motion**

\[ v = |\vec{v}| = \text{constant speed} \]

However, \( \vec{v} \perp \vec{r} \) (radius), or, more precisely,
\( \vec{v} \) along instantaneous tangent
\( \implies \vec{v} \) not constant

- \( T = \text{period} = \text{time for one revolution (rev)} \)

- \( v = \frac{2\pi r}{T} = \text{speed} = \frac{1 \text{ circumference}}{1 \text{ period}} \)

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**Angular Position**

\[ \theta = \frac{s}{r} = \frac{\text{arc length}}{\text{radius}} \]

measures angle in radians (rad)

\[ \theta_{\text{full circle}} = \frac{2\pi r}{r} = 2\pi \text{ rad} = 1 \text{ rev} = 360^\circ \]

\[ 1 \text{ rad} = 1 \text{ rad} \times \frac{360^\circ}{2\pi \text{ rad}} = 57.3 \ldots^\circ \]

SI unit for angle is rad.

Unlike most other units, like m, s, kg, N, etc., rad and \( ^\circ \) (degree) are dimensionless. Being ratios, they are the same in any unit system.
Angular velocity
\[ \Delta \theta = \theta_f - \theta_i \]
\[ \Delta t = t_f - t_i \]

Compare:
\[ \Delta x = x_f - x_i \]

Average angular velocity
\[ \bar{\omega} = \frac{\Delta \theta}{\Delta t} \]

Instantaneous angular velocity
\[ \Delta t \text{ small } \implies \bar{\omega} \text{ becomes } \omega \]

(\(\omega\) is Greek letter omega.)

- \(\omega\) is slope of \(\theta(t)\) graph at \(t\).
- \(\theta_f - \theta_i\) is area under \(\omega(t)\) graph between \(t_i\) and \(t_f\).

Uniform case example:

\[ \theta_f = \theta_i + \omega \Delta t \]
\[ x_f = x_i + v \Delta t \]
Uniform circular motion

\[ \theta_f = \theta_i + \omega \Delta t \]

or \[ \Delta \theta = \theta_f - \theta_i = \omega \Delta t \]

\[ |\Delta \theta| = |\omega| \Delta t = 2\pi \text{ rad} \quad \Longrightarrow \quad \Delta t = T = \text{period} \]

\[ \Longrightarrow \quad \omega T = \pm 2\pi \text{ rad} \quad \Longrightarrow \]

\[ 2\pi \text{ rad} = |\omega| T, \quad |\omega| = \frac{2\pi \text{ rad}}{T}, \quad T = \frac{2\pi \text{ rad}}{|\omega|}. \]

We have three times the same equation! Compare:

\[ F = ma, \quad m = \frac{F}{a}, \quad a = \frac{F}{m}. \]

Thus we need only remember one: \[ |\omega| T = 2\pi \text{ rad}. \]

Radial and tangential components of vector \( \vec{A} \)

\[ A_r = A \cos \phi \]

\[ A_t = A \sin \phi \]
Velocity
\[
\begin{align*}
    v_t &= \frac{\Delta s}{\Delta t} = \frac{r \Delta \theta}{\Delta t} = \omega r, \quad (\omega \text{ in rad/s}) \\
    v_r &= 0 \\
    \Delta \theta &= \omega \Delta t, \\
    \frac{\Delta \theta}{\Delta t} &= \omega, \\
    \frac{r \Delta \theta}{\Delta t} &= \frac{\Delta s}{\Delta t} = v.
\end{align*}
\]

Acceleration?

Break the circular motion up in many small-but-equal time steps $\Delta t$, as in the figure.

Each $\Delta t$ step angle $\theta$ increases by an amount $\Delta \theta$. As $\Delta t \to 0$ the chords $r \Delta \theta$ and arcs $\Delta s$ for each step tend to coincide (i.e. their ratios tend to 1).

Now draw a more detailed figure, also indicating the angles $\alpha$ (Greek letter alpha):
DA and AC both have the same length and direction corresponding to $\Delta \vec{r}_1$, so that CB represents

$$\Delta \vec{r}_2 - \Delta \vec{r}_1 = \vec{v}_2 \Delta t - \vec{v}_1 \Delta t = (\vec{v}_2 - \vec{v}_1) \Delta t$$

$$\implies \Delta \vec{r}_2 - \Delta \vec{r}_1 = \Delta \vec{v} \Delta t$$

(Remark: Velocities $\vec{v}_1$ and $\vec{v}_2$ are averages over the time steps.)

The three isosceles triangles ABC, OAB and ODA are similar, as their three top angles are all $\Delta \theta$. This is easily verified, as for the straight angle at point A and for the three triangles: $\Delta \theta + 2\alpha = 2\pi$ rad = 180°. Hence, we have the proportionality

$$\frac{CB}{AB} = \frac{AB}{AO} \implies |\Delta \vec{v}| \Delta t = \frac{\vec{v} \Delta t}{r}$$

$$\implies \frac{|\Delta \vec{v}|}{\Delta t} = \frac{\vec{v}^2}{r} \implies \vec{a} = \frac{\vec{v}^2}{r}$$

Here $\vec{a}$ and $\vec{v}$ are some averages that tend to instantaneous values when $\Delta t \to 0$, or $\Delta \theta \to 0$ and $\alpha \to 90^\circ$. We thus find that the instantaneous acceleration points to the center.

**Note:** This is one of the hardest derivations in the course. It will not be required on exams. Only the result is needed:
Centripetal acceleration for uniform circular motion:

\[ a_r = \frac{v_t^2}{r} = \omega^2 r \]

\[ a_t = 0 \]

and \( v_t = v \) in this case.

Centripetal force

\[ \vec{F}_{\text{net}} = m\vec{a} = \left( \frac{mv^2}{r}, \text{toward center} \right) \]

= centripetal force, required to stay in circle

In components:

\[ ma_r = (F_{\text{net}})_r = \frac{mv^2}{r} = m\omega^2 r, \]

\[ ma_t = (F_{\text{net}})_t = 0. \]