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1° Momentum conservation in collisions. Application to the completely inelastic collision.

2° Elastic collision: Theory.

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4° Collision in two dimensions.

5° Movie: *Conservation of Momentum: Determination of a Bullet’s Velocity*

   a. To determine the speed of a bullet directly is not easy. A gun is aimed at a wood block attached to a glider that can move frictionless on an airtrack. After the bullet is lodged into the wood, the glider, wood block and bullet move at constant speed. This speed can then be measured with a stopwatch determining the time needed for the glider system to pass two marks a distance \( d \) apart. The experiment is repeated twice and the average \( t \) of the two times is used.

   b. The bullet has mass \( m_1 = 0.0018 \text{ kg} \); the wood block and glider system has mass \( m_2 = 1.9 \text{ kg} \); \( d = 1.0 \text{ m} \); \( t = 3.1 \text{ s} \). The final speed is \( v_f = d/t = 0.32 \text{ m/s} \). Using the principle of conservation of momentum—*In the absence of a net external force the total momentum remains constant*—we have the equation

   \[
   m_1v_{\text{bullet}} + m_20 = (m_1 + m_2)v_f
   \]

   with \( 0 \) the original velocity of glider and woodblock. Putting the numbers in we can solve this easily and we find \( v_{\text{bullet}} = 340 \text{ m/s} = 760 \text{ miles/hour} \).
6° Movie: *Conservation of Momentum: Internal Explosion*

a. Two equal small aluminum cylinders are placed next to one another with a small explosive charge in between. Before the explosion the total momentum \( \mathbf{P} = \sum m \mathbf{v} \) is zero as all velocities are zero. After the explosion the two cylinders move with equal speed but in opposite directions. The total momentum is still zero.

b. The experiment is repeated with one cylinder twice as heavy as the other one. After the explosion the lighter cylinder moves twice as fast, traveling double the distance in the same time. The total momentum is still conserved and remains zero. Indeed, writing \( m_1 = \frac{1}{2} m, \mathbf{v}_{f1} = 2\mathbf{v}, m_2 = m, \mathbf{v}_{f2} = -\mathbf{v} \), we have

\[
\mathbf{P}_f = m_1 \mathbf{v}_{f1} + m_2 \mathbf{v}_{f2} = \frac{1}{2} m \cdot 2\mathbf{v} - m\mathbf{v} = 0.
\]

7° Collision experiments:

a) Drop basketball with small hollow plastic ball on top. After collision with the floor, the small ball takes off going much higher than it was originally when it was dropped. Don’t use a golfball; that may hurt!

b) Use contraption shown in figure: Four balls of decreasing size are restricted to move in one direction only by a straight wire through them. Only the smallest ball can fly off, as the other balls are held together by metal wires. Dropping this “toy” from a small height, oriented as shown in the figure, the little ball really goes high.

c) Newton’s cradle shown again.
Momentum conservation in collisions

Before

\[ \begin{align*}
    m_1 & \quad v_1 = v_{1o} \\
    m_2 & \quad v_2 = v_{2o}
\end{align*} \]

After

\[ \begin{align*}
    m_1 & \quad v_1' = v_{1f} \\
    m_2 & \quad v_2' = v_{2f}
\end{align*} \]

\( \vec{p}_{\text{tot}} \) conserved:

\[ m_1 \vec{v}_1 + m_2 \vec{v}_2 = m_1 \vec{v}_1' + m_2 \vec{v}_2' \]

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**Completely inelastic collision:** \( \vec{v}_1' = \vec{v}_2' = \vec{v}' \)

\[ m_1 \vec{v}_1 + m_2 \vec{v}_2 = (m_1 + m_2) \vec{v}_1', \]

\[ \vec{v}_1' = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}. \]

---

or, with \( \vec{v}_2 = 0, \)

\[ \vec{v}_1 = \frac{m_1 + m_2}{m_1} \vec{v}_1'. \]
Elastic collision

\[ \vec{p}_{\text{tot}}, E_{\text{tot}} \text{ conserved:} \]

\[
\begin{align*}
m_1 \vec{v}_1 + m_2 \vec{v}_2 &= m_1 \vec{v}'_1 + m_2 \vec{v}'_2, \\
\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 &= \frac{1}{2} m_1 v'_1^2 + \frac{1}{2} m_2 v'_2^2
\end{align*}
\]

On a line, “drop arrows” and rearrange:

\[
\begin{align*}
m_1 (v_1 - v'_1) &= m_2 (v'_2 - v_2), \\
\frac{1}{2} m_1 (v_1^2 - v'_1^2) &= \frac{1}{2} m_2 (v'_2^2 - v_2^2)
\end{align*}
\]

Use \[ A^2 - B^2 = (A - B)(A + B) \quad \implies \]

\[
\begin{align*}
m_1 (v_1 - v'_1) (v_1 + v'_1) &= m_2 (v'_2 - v_2) (v'_2 + v_2), \\
m_1 (v_1 - v'_1) &= m_2 (v'_2 - v_2).
\end{align*}
\]

For a real collision, \( v_1 \neq v'_1, v'_2 \neq v_2 \), so we can divide the first equation by the second:

\[
v_1 + v'_1 = v'_2 + v_2
\]

\[ \implies \]

\[
v_1 - v_2 = v'_2 - v'_1
\]

i.e.

Relative velocity flips sign.

(If \( m_2 = \infty \), then \( m_2 (v'_2 - v_2) \) must still remain finite, so that \( v'_2 = v_2 \), like a hard fixed immobile wall, implying \( v'_1 = -v_1 \).)
In summary,

\[ v_1 - v_2 = v'_2 - v'_1, \]
\[ m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2. \]

Ball-on-basketball example:

\[ v'_2 = v_1 - v_2 - v'_1, \]
\[ m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 (v_1 - v_2 - v'_1), \quad \implies \]
\[ (m_1 - m_2) v_1 + 2 m_2 v_2 = (m_1 + m_2) v'_1. \]

Middle picture: \( v_2 = -v_1 \quad \implies \quad \text{final: } v'_1 = \frac{m_1 - 3m_2}{m_1 + m_2} v_1 \)

\[ m_2 \text{ big, } \quad v'_1 \approx -3v_1 \]
\[ 0 = v_F^2 = v_o^2 + 2a(\Delta y) \quad \implies \quad v_o^2 \sim \Delta y \]

\[ h_o = h \quad h_f \approx 3^2 h = 9h \]

The balls drop \( h \) to the floor; the big ball changes direction and collides with the small ball, which then goes up almost \( 9h \).

3
Collision in two dimensions

\[ \vec{p}_{\text{tot},o} = \vec{p}_{\text{tot},f} \]

Original \hspace{2cm} Final

\[ \begin{aligned}
    p_{\text{tot},x} &= m_1 v_1 &= m_1 v'_1 \cos \theta + m_2 v'_2 \cos \phi, \\
    p_{\text{tot},y} &= 0 &= m_1 v'_1 \sin \theta - m_2 v'_2 \sin \phi.
\end{aligned} \]

Elastic \((v_2 = 0)\):

\[ \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v'_1^2 + \frac{1}{2} m_2 v'_2^2. \]

We have 3 equations, with 4 unknowns \((v'_1, v'_2, \theta, \phi)\).

Say \(\theta = 90^\circ, \sin \theta = 1, \cos \theta = 0\):

\[ \begin{aligned}
    m_1 v_1 &= m_2 v'_2 \cos \phi \\
    m_1 v'_1 &= m_2 v'_2 \sin \phi
\end{aligned} \quad \frac{v'_1}{v_1} = \frac{\sin \phi}{\cos \phi} = \tan \phi \]

\[ v'_1 = v_1 \tan \phi \]

\[ v'_2 = \frac{m_1}{m_2} \frac{v'_1}{\sin \phi} = \frac{m_1}{m_2} \frac{v_1}{\cos \phi} \]
Indeed,

\[ m_1 v'_1 = m_2 v'_2 \sin \phi, \]

\[ v'_2 = \frac{m_1 v'_1}{m_2 \sin \phi} = \frac{m_1 v_1}{m_2 \sin \phi} = \frac{m_1 v_1}{m_2 \cos \phi}. \]

Put this into the kinetic energy equation:

\[ \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 (v_1 \tan \phi)^2 + \frac{1}{2} m_2 \left( \frac{m_1 v_1}{m_2 \cos \phi} \right)^2, \]

\[ \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_1^2 \tan^2 \phi + \frac{1}{2} \frac{m_1^2}{m_2} v_1^2 \frac{1}{\cos^2 \phi}. \]

Divide by \( \frac{1}{2} m_1 v_1^2 \), and find

\[ 1 = \tan^2 \phi + \frac{m_1}{m_2} \frac{1}{\cos^2 \phi}, \]

which we can solve for \( \phi \):

\[ \cos^2 \phi = \sin^2 \phi + \frac{m_1}{m_2}, \]

\[ \cos(2\phi) = \cos^2 \phi - \sin^2 \phi = \frac{m_1}{m_2}, \]

\[ \phi = \frac{1}{2} \arccos \frac{m_1}{m_2} \equiv \frac{1}{2} \cos^{-1} \frac{m_1}{m_2}. \]

**Note:** The two-dimensional collision may show up in some homework, but it will not be asked on the exams.