International Journal of Modern Physics B

HAPPER'S CURIOUS DEGENERACIES AND YANGIAN

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Received 27 November 2001

We find raising and lowering operators distinguishing the degenerate states for the Hamiltonian $H = x(K + \frac{1}{2})S_z + \mathbf{K} \cdot \mathbf{S}$ at $x = \pm 1$ for spin 1 that was given by Happer et al.[1,2] to interpret the curious degeneracies of the Zeeman effect for condensed vapor of ^{87}Rb. The operators obey Yangian commutation relations. We show that the curious degeneracies seem to verify the Yangian algebraic structure for quantum tensor space and are consistent with the representation theory of $Y(sl(2))$.

1. Indecomposible Quantum Tensor Space

In Quantum Mechanics, a state is described in terms of wave function, i.e. $|\psi>$ is a vector in Hilbert space. If two particles described by $|\psi_{12}>$ are entangled, there should be "overlapping effect" between V_1 and V_2, i.e., besides V_1 and V_2 we should deal with $V_1 \otimes V_2$, the quantum tensor space. The simplest example is Breit-Rabi's Hamiltonian:

$$H_{\rm BR} = \mathbf{K} \cdot \mathbf{s} + xks_3, \tag{1.1}$$

where $\mathbf{s}$ and $\mathbf{K}$ stand for the spins of electron and atomic nucleus, respectively. $\mathbf{K}^2 = K(K+1)$. On account of the conservation of $\mathbf{K}^2$ and $m = K_3 + s_3$ two independent states are introduced:

$$\alpha_1 >= |K, m - \frac{1}{2} > |\frac{1}{2}, \frac{1}{2} >, \ \ \alpha_2 >= |K, m + \frac{1}{2} > |\frac{1}{2}, -\frac{1}{2} > . \tag{1.2}$$

For a fixed m with the basis $\Phi = \begin{pmatrix} |\alpha_1> \\ |\alpha_2> \end{pmatrix}$, we have

$$H_{\rm BR}^{(m)} = -\frac{1}{4} + \frac{1}{2}[(xk+m)\sigma_3 + \sqrt{k^2-m^2}\sigma_1], \tag{1.3}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are Pauli matrices. Eq. (1.3) can be diagonalized through a rotation:[1]

$$U(\varphi_m)H_{\rm BR}^{(m)}U(\varphi_m)^{-1} = H_{\rm BR}^{(m)}(\varphi_m), \quad \Phi^{(m)}(\varphi_m) = U(\varphi_m)\Phi^{(m)}, \tag{1.4}$$

where

$$\Phi^{(m)}(\varphi_m) = \begin{pmatrix} (\cos\frac{\varphi_m}{2})|\alpha_1> -(\sin\frac{\varphi_m}{2})|\alpha_2> \\ (\sin\frac{\varphi_m}{2})|\alpha_1> +(\cos\frac{\varphi_m}{2})|\alpha_2> \end{pmatrix}, \tag{1.5}$$

$$E = -\frac{1}{4} - \omega_m\sigma_3, \tag{1.6}$$

and

$$\cos\varphi_m = \frac{(xk+m)}{\omega_m}, \quad \omega_m^2 = (1+x^2)k^2 + 2xmk. \tag{1.7}$$

Noting that the rotation angle φ_m is m-dependent and m here cannot be replaced by the operator $K_3 + s_3$. This is because of the nonlinearity in m, i.e., the rotation should depend on the history. Observing Eq. (1.6) and Eq. (1.7) there is not degeneracy for the energy E, because the vanishing ω_m means a complex magnetic field.

However, there appears degeneracies for spin-1 in the experiment.[2] Why the Zeeman effect vanishes at the particular value of applied field? This is the main subject concerned in this paper.

2. Introduction of the Curious Degeneracies

The curious degeneracies observed in the experiment for condensed vapor of ^{87}Rb and ^{85}Rb[1] at 220° under pressure and applied magnetic field $B \sim 1500$ Gauss are converted into "anti-level-crossing" for the triplet ($S = 1$).[1,2] To describe the Hamiltonian of a triplet dimer neglecting the quadrapole interaction, Happer et al. introduced[1,2]

$$H = \mathbf{K}\cdot\mathbf{S} + x(K+\frac{1}{2})S_z, \tag{2.1}$$

and pointed out that when $x = 1$ there appear the curious degeneracies for $S = 1$, where $\mathbf{K}$ and $\mathbf{S}$ are angular momentum and spin, respectively, $\mathbf{K}^2 = K(K+1)$ and $\mathbf{S}^2 = S(S+1)$ with $S = 1$. In Ref. 1, the eigenvectors corresponding to $E = -\frac{1}{2}$ had been given and an elegant discussion was made. However, there remain the following essential questions:

- Why the curious degeneracies occur only for $S = 1$?

- How to distinguish the degenerate states?

We would like to present the answer in this paper.

For $x = \pm 1$, the eigenequation

$$H\Psi_m = E_m \Psi_m \tag{2.2}$$

has three types of solutions whose eigenstates are denoted by α_T, α_D and α_B with the corresponding energies $E_T > E_D > E_B$, respectively. For the D-set, $H\alpha_{Dm} = -\frac{1}{2}\alpha_{Dm}$, there appear the curious degeneracies called Happer degeneracies that has been supported by the experiment.[2] The results of Happer can be summarized in the Table 1 ($\mathbf{G} = \mathbf{K} + \mathbf{S}$, $G_3 = m$).

	$G=$ $K+1$	$G=$ K	$G=$ $K-1$		$D-$set	$T-$set	$B-$set
$K+1$	$--$			$\rightarrow$		$\alpha_{T,m=K+1}$	
K	$--$	$--$		$\rightarrow$	$\alpha_{D,m=K}$	$\alpha_{T,m=K}$	
$K-1$	$--$	$--$	$--$	$\rightarrow$	$\alpha_{D,m=K-1}$	$\alpha_{T,m=K-1}$	$\alpha_{B,m=K-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	$--$	$--$	$--$	$\rightarrow$	α_{Dm}	α_{Tm}	α_{Bm}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$-K+1$	$--$	$--$	$--$	$\rightarrow$	$\alpha_{D,m=-K+1}$	$\alpha_{T,m=-K+1}$	$\alpha_{B,m=-K+1}$
$-K$	$--$	$--$		$\rightarrow$		$\alpha_{T,m=-K}$	$\alpha_{B,m=-K}$
$-K-1$	$--$			$\rightarrow$	$\alpha_{D,m=-K-1}$		

Table 1

We emphasize that the states with $m = K+1$ and $m = -K$ for $x = 1$ ($m = -K-1$ and $m = K$ for $x = -1$) in the D-set are excluded. For simplicity we discuss the case for $x = 1$ henceforth. The eigenstates of H are linear combinations of the states of $G = K+1, K$ and $K-1$. Since the shortage of states with $m = K+1$ and $m = -K$ it is not surprise to appear the unusual thing to distinguish the m-dependent states, for example in Eq. (1.6).

3. Yangian as the Raising and Lowering Operator for the Degenerate States

Let us first recall how to establish the Lie algebraic structure in Quantum Mechanics. For the given $(2K+1)$ states denoted by $|K, K_3 = K>$, $|K, K_3 = K-1>, \cdots$, and $|K, K_3 = -K>$, the raising (or lowering) operator K_+ (or K_-) can be introduced such that for any $m = K_3$,

$$K_\pm |K, m> \sim |K, m \pm 1>, \tag{3.1}$$

and $K \pm |K, \pm K>= 0$. Through checking the commutation relations for $K_\pm$ and K_3, we say that the Lie algebraic structure is found if the commutation relations are closed. It is emphasized that there is not m-dependence in the operators $K_\pm$ in Eq.

(3.1), because the eigenvalues of K_3 are uniform. However, suppose the eigenvalues are not uniform, the raising and lowering operators should depend on m, i.e., it should indicate on which state the operators act. Actually, such "starting state" dependence occurs more often in nonlinear models.[3]

After calculations, we have found the raising operator for the D-set in the table 1 (at $x = \pm 1$):

$$J_+ = (m + K + 1)G_+ + j_+(a, b), \tag{3.2}$$

where

$$j_+(a, b) = aS_+ + bK_+ + \frac{1}{2}(S_3K_+ - S_+K_3), \tag{3.3}$$

and

$$a = -\frac{K}{2}, b - a = \frac{1}{2}(K + 1), G_+ = K_+ + S_+. \tag{3.4}$$

Noting that $(b - a)$ is independent of m. Whereas

$$J_- = -(m + K)G_- + j'_-(c, d), \tag{3.5}$$

where

$$j'_-(c, d) = cS_- + dK_- - \frac{1}{2}(S_3K_- - S_-K_3), \tag{3.6}$$

and

$$c = \frac{K}{2} + \frac{1}{2}, d - c = -\frac{K}{2}, G_- = K_- + S_-. \tag{3.7}$$

It can be checked that for $x = 1$, $J_+|\alpha_{D,m=K} >= 0$ and $J_+|\alpha_{D,m=-K-1} >= 0$.

Obviously the $J_\pm$ shown in Eq. (3.2) and Eq. (3.5) are special form of the Yangian operator:

$$\mathbf{J} = \lambda\mathbf{G} + \mathbf{j}, \tag{3.8}$$

where

$$\mathbf{j} = \mu\mathbf{K} + \gamma\mathbf{S} - \frac{i}{2}\mathbf{S} \times \mathbf{K}, \tag{3.9}$$

and λ, μ, γ are arbitrary constants. A set formed by both $\mathbf{J}$ and $\mathbf{j}$ satisfy $Y(sl(2))$ defined by Drinfeld,[4] and is related to the Yang-Baxter equations.[5,6]

4. Yangian Algebra

The commutation relations for $\mathbf{J}$ and the total angular momentum $\mathbf{I} = \mathbf{G} = \mathbf{S} + \mathbf{K}$ form the so-called Yangian algebra associated with $sl(2)$. The parameters μ and γ play the important role in the representation theory of Yangian given by Chari and Pressley.[7] Many chain models possess the Yangian symmetry, for example, for 1-d Hubbard model and Haldane-Shastry model.[8] The set $\{\mathbf{I},\ \mathbf{J}\} = Y(sl(2))$ obeys the commutation relations of $Y(sl(2))$ ($A_\pm = A_1 \pm \sqrt{-1}A_2$):

$$[I_3,\ I_\pm] = \pm I_\pm,\ \ [I_+,\ I_-] = 2I_3,\quad (sl(2)); \tag{4.1}$$

$$[I_3,\ J_\pm] = [J_3,\ I_\pm] = \pm J_\pm,\ \ [I_+,\ J_-] = [J_+,\ I_-] = 2J_3, \tag{4.2}$$

(i.e. $[I_i,\ J_j] = \sqrt{-1}\varepsilon_{ijk}J_k$) and nonlinear relation

$$[J_3,\ [J_+,\ J_-]] = \frac{1}{4}I_3(I_+J_- - J_+I_-) \tag{4.3}$$

that forms an infinitely dimensional algebra. All the other relations given in Ref. 4 can be obtained from Eq. (4.1)–Eq. (4.3) together with the Jacobian identities.[9,10]

The essential difference between the representations of Yangian algebras and those of Lie algebras is the appearance of the free parameters μ and γ whose originally physical meaning is one-dimensional momentum. Their special choice specifies a particular model. Applying the Yangian representation theory to Hydrogen atom, it yields the correct spectrum ($\sim n^{-2}$) that is the simplest example of the application of Yangian in Quantum Mechanics.[10] Now the Happer's degeneracies can be viewed as another example. Furthermore, we would like to make the following remarks:

(a) The elements of J_+ given by Eq. (3.2)

$$< \alpha_{Dm'}|J_+|\alpha_{Dm} > \sim < \alpha_{Dm'}|K_+|\alpha_{Dm} > \neq 0,$$

because $< \alpha_{Dm'}|\mathbf{S}|\alpha_{Dm} > = < \alpha_{Dm'}|\mathbf{S}\times\mathbf{K}|\alpha_{Dm} > = 0$, as pointed out in Ref. 1 (see Eq. (2.23) in Ref. 1). This indicates that the role played by J_+ in the "D-direction" is like that played by K_+. Why do we need a Yangian? The terms of S_+ and $(\mathbf{K}\times\mathbf{S})_+$ should be added to guarantee $< \alpha_{Tm'}|J_+|\alpha_{Dm} > = < \alpha_{Bm'}|J_+|\alpha_{Dm} > = 0$, namely, if only acting K_+ on α_{Dm} it yields non-vanishing transitions to $\alpha_{Tm'}$ and $\alpha_{Bm'}$ that no longer preserves the D-set. The part other than K_+ in the Yangian J_+ given by Eq. (3.2) exactly cancel the nonvanishing contribution received from "T-" and "B-direction".

(b) Observing the process determining parameters a and b in Eq. (3.3), the reason for the existence of solution of a and b is clear. For $S = 1$, the eigenvector of H is formed by three base. Apart of an over-all normalization factor there are two independent coefficients. In requiring $J_+\alpha_{Dm} \sim \alpha_{Dm+1}$, we have to compare the coefficients of the independent base in $J_+\alpha_{Dm}$ and α_{Dm+1} to determine the unknown parameters a and b. For spin $S = 1$, there are just two equations for a and b. However, for spin $S > 1$, in general, one is unable to find solution for a and b to fit more than two equations. Therefore, the Yangian description of the curious degeneracies admits only $S = 1$ for arbitrary K. This is consistent with experiment.[1,2]

(c) In fact, the parameters appearing in J_+ and J_- exactly coincide with the conditions of the existence of the subrepresentations of the Yangian.[7] Following the theorem in Ref. 7, for $a - b = -\frac{K}{2} - \frac{1}{2}$ the subspace spanned by vectors with $G = K + 1$ is the unique irreducible subrepresentation of $Y(sl(2))$, that is, the states with $G = K + 1$ are stable under the action of $\mathbf{J}$. Note that the existence and uniqueness of subrepresentation is only related to the difference of a and b. Moreover, for the given a and b in Eq. (3.4), the action of J_+ on the states with

$G = K + 1$ is given by $J_+\alpha_{G=K+1,m} = (m + K + 1)G_+\alpha_{G=K+1,m}$ and at the same time, J_+ will make the states with $G = K$ and $G = K - 1$ transit to $G = K + 1$, but not vice versa, called "directional transition",[9] i.e. the transition given rise by Yangian goes in one way. Thus, for the given a and b in Eq. (3.4), the set of states with $G = K + 1$ and D-set are stable under the action of J_+ simultaneously. For $c - d = \frac{K}{2}$, $G = K - 1$ is the unique irreducible subrepresentation and for c and d given by Eq. (3.7), acting J_- on the states with $G = K - 1$, we have $J_-\alpha_{G=K-1,m} = -(m + K)G_-\alpha_{G=K-1,m}$. Therefore the representation theory of $Y(sl(2))$ tells that the relationship between $a - b$ and $c - d$ given by Eq. (3.4) and Eq. (3.7), respectively, should be held to preserve the states with $G = K + 1$ (or $G = K - 1$) that possesses Lie algebraic behavior.

(d) We have seen that the J_- is not the conjugate of J_+. Such a phenomenon is reasonable because α_{Dm} is neither the Lie-algebraic state nor symmetry of H. In fact, if α is not an eigenstate of $\mathbf{I}^2$ ($\mathbf{I}$ belongs to a Lie algebra) and $I_+\alpha \sim \alpha_1$, we cannot have $I_-\alpha_1 \sim \alpha$. Now there is the similarity for Yangian. Moreover, the D-set is not a subrepresentation of $Y(sl(2))$, i.e., D-set cannot be stable under all the actions of $\mathbf{J}$, but stable under J_+ and J_- with the different parameters which just satisfy the condition for subrepresentation of Yangian.

(e) The third component of $\mathbf{J}$ takes the form $J_3 = aS_z + bK_z + S_+K_- - S_-K_+$. For any parameters, the action of J_3 will not keep the D-set. But, with the suitable $a - b = 1$, the operator $J_3 + 2(2K + 1)S_z^2$ will keep the D-set.

(f) We emphasized that the m appearing in Eq. (3.2) and Eq. (3.5) cannot be replaced by the operator G_3. It appears as a parameter in Yangian. The m-dependents only indicates that the raising or lowering operation depends on "history" in difference from the Lie algebraic structure.

In conclusion we have read of a new type of algebra structure(Yangian) from the Happer's degeneracies and such an algebra had been ready by Drinfeld.[4] All the analysis coincides with the representation theory of $Y(sl(2))$[7] for the special choice of a, b in J_+ and c, d in J_-. It also leads to the fact that only $S = 1$ is allowed to yield the curious degeneracies. If the Zeeman effect tells Lie algebra, then the curious degeneracies possibly tell the existence of Yangian.

Acknowledgements

We would like to thank Freeman Dyson who guided the authors to deal with this problem and W. Happer for encouragement and enlightening communications. This work is in part supported by NSF of China.

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