Department of Physics

Preliminary Examination

Classical Mechanics

January 2, 1989 9:00 a.m. to 12:00 Noon

Instructions:

1. Calculators are permitted. Write the answer to each question on a separate sheet of paper. If more than one sheet is required, staple all the sheets corresponding to a single question together. Be sure your identification number is written on each problem sheet.

2. All questions carry the same credit and each should receive approximately the same amount of your exam time.

1. A large uniform box of mass M whose height is three times the width of its square base (h = 3w) rests on a plane which may be raised through an angle Θ as shown in the figure 1. The coefficients of kinetic and static friction are such that $\mu_k = 0.5\mu_s$. (Let $\mu_k \equiv \mu$ and $\mu_s \equiv 2\mu$.) In considering the equilibrium and the motion of the box as the angle Θ is increased from 0, include free body diagrams indicating the magnitude, direction, and line of action of each force acting on the box.

a. Determine the minimum value of μ for which the box will tip rather than slide and find the angle Θ_o at which the box will tip.

b. For the range of μ for which the block will slide down the plane determine the linear acceleration of the block as a function of μ at the angle for which the box just begins to slide.

2. A Yo–Yo^{tm} of mass M and moment of inertia I about an axis through its center of mass is pulled by a force F on the cord as shown in the figure 2.

a. Write down the equations of motion for the Yo–Yo^{tm}, remembering that Newton's laws are valid only in an inertial reference frame.

b. If the Yo–Yo^{tm} rolls without slipping, find the acceleration of the center of mass of the Yo–Yo^{tm}.

c. In what direction does the Yo–Yo^{tm} move?

d. With μ_s and μ_k the static and kinetic coefficients of friction respectively, determine any restriction(s) on the force F for the Yo–Yo^{tm} to avoid slipping.

3. Consider two reference frames as shown in the figure 3. One an inertial frame with origin at the center of the earth (whose orbital motion about the sun is ignored) and axes fixed with respect to the distant stars and a second whose origin is at the surface of the earth with axes fixed with respect to the terrain.

Recall (hopefully) that if $\vec{A}$ is any vector quantity (position, velocity, etc.) the time rate of change of $\vec{A}$ as measured by an observer at rest with respect to the earth–fixed (*) frame and the time rate of change $\vec{A}$ as measured by an observer at rest with respect to the inertial frame are related by

$$\frac{d}{dt}\vec{A} = \vec{\omega} \times \vec{A} + \frac{d^*}{dt}\vec{A}.$$

where $\vec{\omega}$ is the angular velocity of the (*) frame with respect to the inertial frame.

a. Beginning with Newton's second law valid in the inertial reference frame show that the equation of motion for a particle of mass m valid in the non–inertial (*) frame is that

$$m \frac{d^{*2}}{dt^2} \vec{r} = \sum_{\alpha} \vec{F}_{\alpha} - \{ 2m\vec{\omega} \times \frac{d^{*}}{dt} \vec{r} + m\vec{\omega} \times (\vec{\omega} \times [\vec{R} + \vec{r}])\}$$

where the summation is over the "real" external forces (e.g. gravity). $\vec{R}$ is a vector from the center of the earth to the origin of the (*) frame and $\vec{r}$ is the vector from the origin of the (*) frame to the position of the particle.

b. Identify the "coriolis force" term.

c. Choose a coordinate system for the non–inertial (*) frame with orthogonal unit vectors $\hat{e}$, $\hat{n}$, $\hat{u}$ (east, north, up). With Θ the latitude angle, show that the angular velocity vector $\vec{\omega}$ is represented in this basis as

$$\vec{\omega} = (\omega\cos\Theta)\,\hat{n} + (\omega\sin\Theta)\,\hat{u}.$$

d. Estimate the deflection of a mass initially traveling due north at 100 m/s over a distance of 1 kilometer.

4a. Construct the Lagrangian and find an equation of motion for a bead of mass m sliding freely on a smooth, circular hoop of radius b which rotates in a horizonal plane about one of its points O with constant angular velocity ω. (See the figure 4, a generalized coordinate has been suggested.)

b. What is the frequency of small (amplitude) oscillations in Θ?

5. A nuclear collision process is found to have the following kinematics: An incident deuteron with kinetic energy $T_{di} = 1.808$ MeV strikes a second deuteron at rest producing a proton with kinetic energy $T_{pf} = 3.476$ MeV which goes off at right angles to the direction of the incident deuteron and a triton whose kinetic energy and momentum are not measured. (All quantities given are measured in the lab frame.) The known rest mass energies of the proton and deuteron are:
$m_{op}c^2 = 938.3$ MeV and $m_{od}c^2 = 1875.6$ MeV. (See the figure 5)

a. Write down the energy–momentum conservation laws.

b. Show that the rest mass energy $m_{ot}c^2$ of the triton is given by

$$(m_{ot}c^2)^2 = (m_{op}c^2)^2 + 2(2m_{od}c^2 + T_{di})\,([m_{od} - m_{op}]c^2 - T_{pf})$$

c. From the information given determine the rest mass energy of the triton.

6. A particle of mass m moves in one dimension under the influence of the potential energy function

$$V(x) = V_o \sin^2(x/a).$$

a. Sketch the phase space trajectory of the particle for the initial conditions $x(O) = O$ and $p(O) = p_o$ when

1). $\frac{p_o^2}{2m} > V_o$; 2). $\frac{p_o^2}{2m} < V_o$.

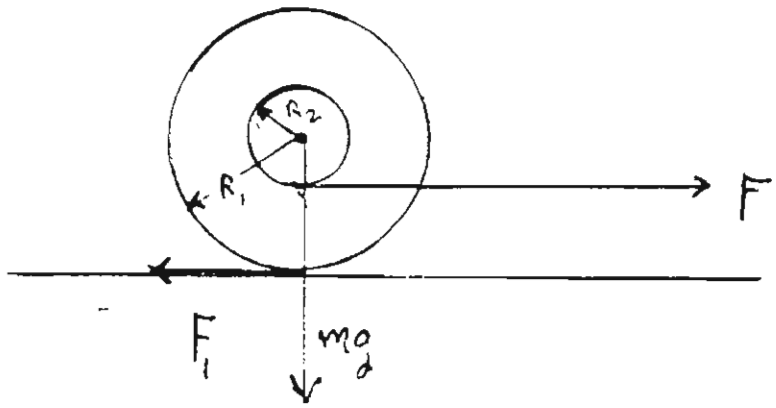
b. Derive an expression for the period τ for closed trajectories of the particle.

c. Find the period τ of the motion for $\frac{p_o^2}{2m} = \frac{1}{2}V_o$.

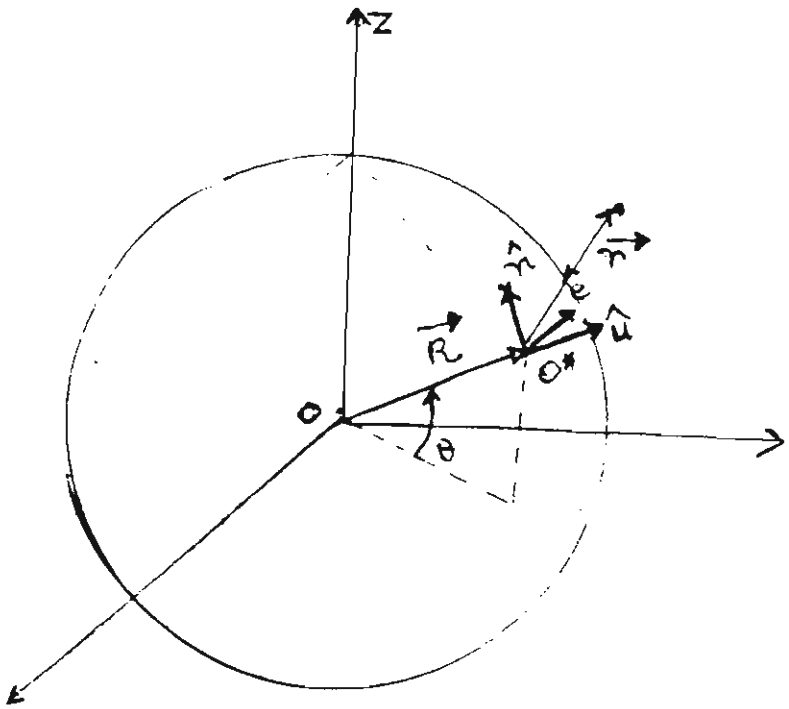
d. Find the approximate period of the motion when $\frac{p_o^2}{2m} <<< V_o$.

Fig. 1

II



II



Fig

Iv

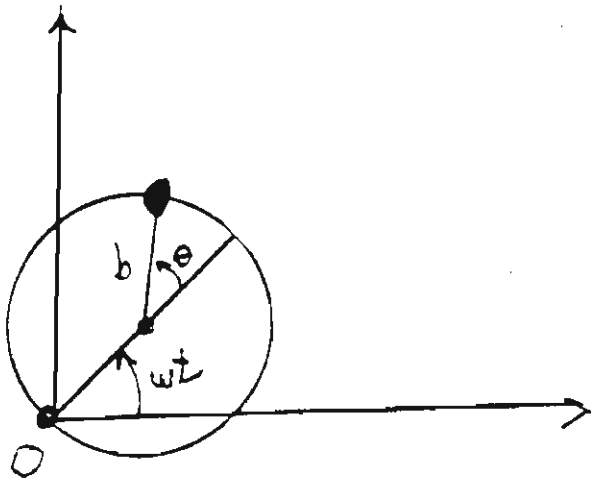
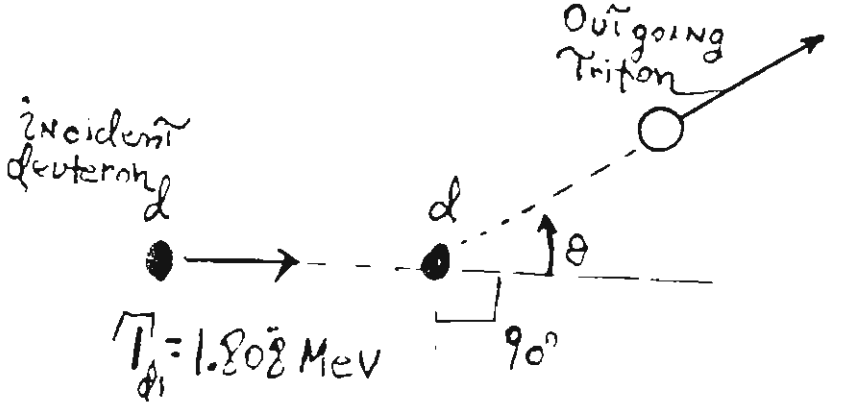
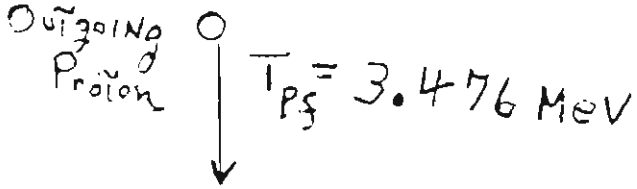


FIG 4

V.


Outgoing Triton
Incident deuteron
d
d
θ
90°
$T_{d_1} = 1.808$ MeV


Outgoing Proton
$T_{P_f} = 3.476$ MeV

COMPLETE ELLIPTIC INTEGRALS

$$K = \int_0^{\pi/2} \frac{d\Phi}{\sqrt{1 - k^2 \sin^2 \Phi}}$$

$$E = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \Phi}\, d\Phi = E\left(k, \frac{\pi}{2}\right)$$

$\sin^{-1} k$	K	$\log K$	$\sin^{-1} k$	K	$\log K$
0°	1.5708	0.196120	45°	1.7568	0.252065
1	1.5709	0.196153	46	1.7892	0.252588
2	1.5713	0.196252	47	1.8122	0.258197
3	1.5719	0.196418	48	1.8256	0.261406
4	1.5727	0.196649	49	1.8396	0.264716
5	1.5738	0.196947	50	1.8541	0.268127
6	1.5751	0.197312	46	1.8691	0.271644
7	1.5767	0.197743	47	1.8548	0.275267
8	1.5785	0.198241	48	1.9011	0.279001
9	1.5805	0.198806	49	1.9180	0.282848
10	1.5828	0.199438	50	1.9356	0.286811
11	1.5854	0.200137	51	1.9539	0.290895
12	1.5882	0.200904	52	1.9729	0.295101
13	1.5913	0.201740	53	1.9927	0.299438
14	1.5946	0.202643	54	2.0133	0.303901
15	1.5981	0.203615	55	2.0347	0.308504
16	1.6020	0.204657	56	2.0571	0.313247
17	1.6061	0.205768	57	2.0804	0.318138
18	1.6105	0.206945	58	2.1047	0.323182
19	1.6151	0.208200	59	2.1300	0.328384
20	1.6200	0.209522	60	2.1565	0.333753
21	1.6252	0.210916	61	2.1842	0.339295
22	1.6307	0.212382	62	2.2132	0.345020
23	1.6365	0.213921	63	2.2435	0.350936
24	1.6426	0.215533	64	2.2754	0.357053
25	1.6490	0.217219	65	2.3088	0.363384
26	1.6557	0.218981	66	2.3439	0.369940
27	1.6627	0.220818	67	2.3809	0.376736
28	1.6701	0.222732	68	2.4198	0.383787
29	1.6777	0.224723	69	2.4610	0.391112
30	1.6858	0.226793	70	2.5046	0.398730
31	1.6941	0.228943	71	2.5507	0.406665
32	1.7028	0.231173	72	2.5998	0.414943
33	1.7119	0.233485	73	2.6521	0.423596
34	1.7214	0.235880	74	2.7081	0.432650
35	1.7312	0.238359	75	2.7681	0.442176
36	1.7415	0.240923	76	2.8327	0.452196
37	1.7522	0.243575	77	2.9026	0.462792
38	1.7633	0.246315	78	2.9786	0.474006
39	1.7748	0.249146	79	3.0617	0.485967
40	1.7868	0.252068	80	3.1534	0.498777

COMPLETE ELLIPTIC INTEGRALS (Continued)

$\sin^{-1} k$	K	$\log K$	$\sin^{-1} k$	K	$\log K$
80°	3.1534	0.498777	85°	4.0317	0.605396
81	3.2553	0.512591	86	4.2095	0.624355
82	3.3699	0.527513	87	4.4387	0.647255
83	3.5004	0.544120	88	4.7427	0.676027
84	3.6519	0.562514	89	5.1349	0.710192
85	3.8317	0.583396	90	∞	∞

Values of K for $\sin^{-1} k$ = 85° to 90° by 0.1° and 89° to 90° by minutes.

$\sin^{-1} k$	K	$\log K$	$\sin^{-1} k$	K	$\log K$
85.0°	3.832	0.58343	89° 0′	5.435	0.73520
85.1	3.852	0.58569	89 2	5.469	0.73791
85.2	3.872	0.58794	89 4	5.504	0.74068
85.3	3.893	0.59028	89 6	5.540	0.74351
85.4	3.914	0.59262	89 8	5.578	0.74648
85.5	3.936	0.59506	89 10	5.617	0.74950
85.6	3.958	0.59748	89 12	5.658	0.75268
85.7	3.981	0.59999	89 14	5.700	0.75597
85.8	4.004	0.60249	89 16	5.745	0.75939
85.9	4.028	0.60509	89 18	5.791	0.76275
86.0	4.053	0.60778	89 20	5.840	0.76641
86.1	4.078	0.61045	89 22	5.891	0.77019
86.2	4.104	0.61321	89 24	5.946	0.77422
86.3	4.130	0.61595	89 26	6.003	0.77837
86.4	4.157	0.61878	89 28	6.063	0.78259
86.5	4.185	0.62170	89 30	6.128	0.78732
86.6	4.214	0.62469	89 32	6.197	0.79218
86.7	4.244	0.62778	89 34	6.271	0.79734
86.8	4.274	0.63093	89 36	6.351	0.80284
86.9	4.306	0.63407	89 38	6.438	0.80875
87.0	4.339	0.63739	89 40	6.533	0.81511
87.1	4.372	0.64068	89 41	6.584	0.81849
87.2	4.407	0.64414	89 42	6.639	0.82210
87.3	4.444	0.64777	89 43	6.696	0.82582
87.4	4.481	0.65137	89 44	6.756	0.82969
87.5	4.520	0.65514	89 45	6.821	0.83385
87.6	4.562	0.65916	89 46	6.890	0.83822
87.7	4.603	0.66304	89 47	6.964	0.84286
87.8	4.648	0.66727	89 48	7.044	0.84792
87.9	4.694	0.67154	89 49	7.131	0.85315
88.0	4.743	0.67605	89 50	7.226	0.85890
88.1	4.794	0.68070	89 51	7.332	0.86522
88.2	4.848	0.68556	89 52	7.449	0.87210
88.3	4.905	0.69064	89 53	7.583	0.87984
88.4	4.965	0.69592	89 54	7.737	0.88857
88.5	5.030	0.70157	89 55	7.919	0.89867
88.6	5.099	0.70749	89 56	8.143	0.91078
88.7	5.173	0.71374	89 57	8.430	0.92583
88.8	5.253	0.72041	89 58	8.836	0.94626
88.9	5.340	0.72754	89 59	9.529	0.97905
89.0	5.435	0.73520	90 0	∞	∞

FORMS CONTAINING $\sqrt{a^2 - u^2}$

27. $$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C.$$

28. $$\int \sqrt{a^2 - u^2}\, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a} + C.$$

29. $$\int (a^2 - u^2)^{\frac{n}{2}} du = \frac{u(a^2 - u^2)^{\frac{n}{2}}}{n+1} + \frac{a^2 n}{n+1} \int (a^2 - u^2)^{\frac{n}{2}-1} du.$$

30. $$\int u^2 \sqrt{a^2 - u^2}\, du = -\frac{u}{4}(a^2 - u^2)^{\frac{3}{2}} + \frac{a^2}{8} u\sqrt{a^2 - u^2} + \frac{a^4}{8} \sin^{-1} \frac{u}{a} + C.$$

31. $$\int u^m (a^2 - u^2)^{\frac{n}{2}} du = -\frac{u^{m-1}(a^2 - u^2)^{\frac{n}{2}+1}}{n+m+1} + \frac{a^2(m-1)}{n+m+1} \int u^{m-2}(a^2 - u^2)^{\frac{n}{2}} du$$

32. $$\int \frac{du}{(a^2 - u^2)^{\frac{n}{2}}} = \frac{u}{a^2\sqrt{a^2 - u^2}} + C.$$

33. $$\int \frac{u^2 du}{\sqrt{a^2 - u^2}} = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a} + C.$$

34. $$\int \frac{u^m du}{\sqrt{a^2 - u^2}} = -\frac{u^{m-1}\sqrt{a^2 - u^2}}{m} + \frac{(m-1)a^2}{m} \int \frac{u^{m-2} du}{\sqrt{a^2 - u^2}}.$$

35. $$\int \frac{u^2 du}{(a^2 - u^2)^{\frac{3}{2}}} = \frac{u}{\sqrt{a^2 - u^2}} - \sin^{-1} \frac{u}{a} + C.$$

36. $$\int \frac{u^m du}{(a^2 - u^2)^{\frac{n}{2}}} = -\frac{u^{m-1}}{(m - n + 1)(a^2 - u^2)^{\frac{n}{2}-1}} + \frac{a^2(m-1)}{m - n + 1} \int \frac{u^{m-2} du}{(a^2 - u^2)^{\frac{n}{2}}}$$

37. $$\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a} \ln \frac{a + \sqrt{a^2 - u^2}}{u} + C.$$

38. $$\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{\sqrt{a^2 - u^2}}{a^2 u} + C.$$