Department of Physics

Preliminary Examination

Quantum Mechanics and Modern Physics

Thursday, January 9, 1997

9:00 AM – noon

Instructions:

1. Write your answer to each question on a separate sheet of paper. If more than one sheet is needed staple all of the pages corresponding to a *single* question together in the correct order. But, *do not* staple all of the problems together. This exam has 6 questions.
2. Be sure to write your identification number on each problem sheet.
3. The time allowed for this exam is three hours. All questions carry the same amount of credit. Manage your time carefully.
4. If a question has more than one part, it may not be necessary to successfully complete one part to do the other parts.
5. The exam will be evaluated, in part, by such things as the clarity and organization of your responses. It is a good idea to use short written explanatory statements between the lines of a derivation, for example. Be sure to substantiate any answer by calculations or arguments as appropriate. Be complete, explicit, and concise.
6. The use of electronic calculators and Integral Tables are permitted. However obtaining pre-programmed information from programmable calculators or using any other reference material is strictly prohibited. Oklahoma State University Policies and Procedures on Academic Dishonesty will be followed.

Constants:

$e^- = 1.6\times10^{-19}$ C

$m_e = 9.1\times10^{-31}$ Kg

$\varepsilon_o = 8.85\times10^{-12}\ C^2/(N\cdot m^2)$

$\hbar = 1.055\times10^{-34}$ J·s

1 eV = 1.6×10^{-19} J

HELP:

$$\sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$\cosh(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\mathrm{sech}(x) = \frac{1}{\cosh(x)}$$

$$\mathrm{sech}^2(x) + \tanh^2(x) = 1$$

$$\frac{d}{dx}\mathrm{sech}(x) = -\mathrm{sech}(x)\tanh(x)$$

$$\frac{d}{dx}\tanh(x) = \mathrm{sech}^2(x)$$

$$\int \mathrm{sech}^2(x)dx = \tanh(x)$$

$$\int_{-\infty}^{\infty} x^2\,\mathrm{sech}^2(x)dx = \frac{\pi^2}{6}$$

$$\int \mathrm{sech}^4(x)dx = \tanh(x) - \frac{1}{3}\tanh^3(x)$$

1. a) List the four principles used by Bohr in developing his model for the hydrogen atom.

   b) Use these principles in order to determine general expressions for the allowed energy levels and the orbital radii of single electron atoms or ions.

   c) In a muonic atom or ion, the single electron is replaced by a muon which has the same charge but a mass 207 times that of an electron. Calculate the radii and energies of the ground and first excited states of a muonic helium ion (He^+).

2. a) Define Hilbert space.

b) Associated with a particular system are the Hamiltonian H and some operator M, with the commutation relation $[H,M]=CM$ where C is a constant. If ψ is an eigenstate of H with energy E,

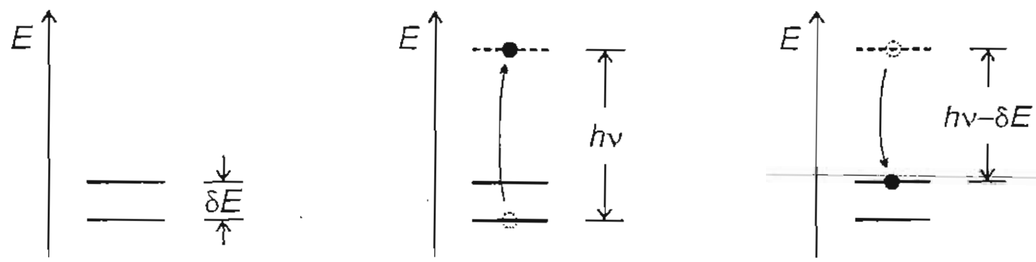
(1) show that $M\psi$ is also an eigenstate of H

(2) find the energy of this state.

c) Show that a Hermitian operator has real eigenvalues.

d) Show that if A and B are two operators that commute, and if there are no degeneracies, then the eigenfunctions of A are also eigenfunctions of B.

3. Consider a molecule which contains two closely-spaced energy levels, as shown in the first figure below. (Assume that any other levels are far removed.) An electron in one of the two levels can absorb an incident photon, thus promoting it to a *virtual level* at an energy $h\nu$ above the original level, where $h\nu$ is the photon energy. This process is depicted in the second figure. Subsequently, the electron can emit a photon and relax to the *other* level, as shown in the third figure. The energy of the emitted photon is shifted by an amount equal to the energy difference between the two levels.

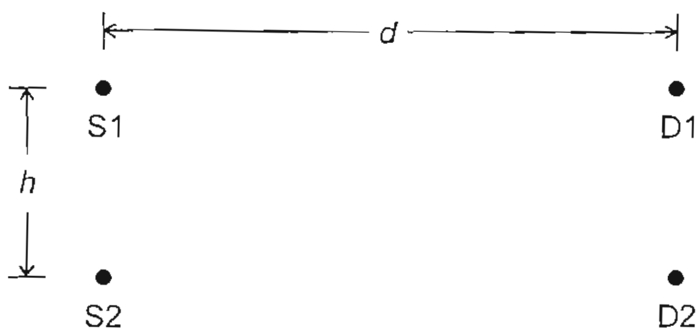


photon absorption   photon emission

a) Since the virtual level is not a real electronic state, this process violates energy conservation, at least temporarily. The uncertainty principle allows a system to violate the various conservation relations within certain restrictions. Assuming that a typical photon has an energy of 1 eV, and that $\delta E \ll h\nu$, what is the time scale of this interaction? In other words, how long can the electron remain in the virtual state before emitting a photon?

b) Explain (qualitatively) how this process can lend a quantum mechanical interpretation to the phenomenon of the index of refraction (that is, to the observation that electromagnetic radiation propagates more slowly through a material medium than it does through vacuum).

4. In any experiment, if a given outcome (i.e., the result of a "measurement") can occur via two or more indistinguishable paths, interference will occur. Consider the following experimental arrangement, where two sources (S1 and S2) emit photons which may be detected by two detectors (D1 and D2). The two sources are arranged so that at some random time, each source emits a single photon. The two photons are identical, having the same frequency and phase. Most of the time, zero or one of these photons is detected—we will discard these measurements. Occasionally, both photons are detected close enough together in time that we can reasonably infer that they constitute the members of a simultaneously-emitted pair. There are three possible outcomes: (1) both photons are detected by D1, (2) both photons are detected by D2, and (3) one photon is detected by D1 and one by D2. We will label the probabilities of these three possible outcomes p_{11}, p_{22}, and p_{12}, respectively, with $p_{11} + p_{12} + p_{22} = 1$.



a) Give expressions for the three probabilities, in terms of the distances d and h and the photon frequency ω. Assume that the photons travel in straight lines. (Hint: The probability amplitude for a photon emitted at a point r_1 to be detected at a point r_2 is proportional to $e^{i\omega t}/t$, where t is the time interval between emission and detection.)

b) Assuming that $h << d$, expand p_{12}, neglecting higher-order terms. Show that p_{12} varies sinusoidally as h is varied, thus demonstrating interference.

Possibly useful relations:

$$\frac{1}{1+x} \approx 1 - x, \quad x << 1 \qquad\qquad \sqrt{1+x} \approx 1 + \frac{x}{2}, \quad x << 1$$

5. Consider the system consisting of a particle of mass m in the finite potential well given by $V(x) = -\frac{\hbar^2 a^2}{m}\text{sech}^2(ax)$, where a is a constant that determines the depth and width of the well. The system has an energy eigenstate with energy $E = -\frac{\hbar^2 a^2}{2m}$ and the following wave function:

$$\psi(x) = \sqrt{\frac{a}{2}}\,\text{sech}(ax).$$

Like the Gaussian, the hyperbolic secant is a self-transformed function; that is, its Fourier transform is also a hyperbolic secant. Thus the corresponding momentum eigenfunction is

$$\varphi(k) = \sqrt{\frac{\pi}{4a}}\,\text{sech}\left(\frac{\pi k}{2a}\right).$$

Unlike the Gaussian, however, the hyperbolic secant is not a minimum-uncertainty wave function; the position-momentum uncertainty product satisfies $\Delta x \Delta p > \hbar/2$.

(a) Verify that $\psi(x)$ and $\varphi(k)$ are properly normalized. Does $\psi(x)$ represent the ground state or an excited state? Explain how you can tell.

(b) Calculate the position-momentum uncertainty product and compare to its minimum possible value.

6. An attractive delta-function potential of strength $aV_0 > 0$ is located at the center of an infinite square well of width a. The Hamiltonian for a particle of mass m moving in this potential is

$$\mathsf{H} = \begin{cases} \dfrac{\mathbf{p}^2}{2m} - aV_0\delta(x), \ldots\ldots |x| \le a/2 \\ \infty, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots |x| > a/2 \end{cases}.$$

(Note that for small enough V_0, the delta function can be treated as a perturbation.)

(a) Show, by integrating the Schrödinger equation, that because of the delta function the wave functions in general undergo a discontinuous change of slope at $x = 0$:

$$\left.\frac{d\psi}{dx}\right|_{0^+} - \left.\frac{d\psi}{dx}\right|_{0^-} = -\frac{2maV_0}{\hbar^2}\psi(0).$$

(b) For $V_0 = 0$, write the normalized energy eigenfunctions. For $V_0 \neq 0$ (not necessarily small), sketch qualitatively the wave functions belonging to the lowest three energy levels.

(c). For small V_0, treat the delta function as a perturbation and find the first-order correction to the energy for each of the lowest three states. If any of these first-order corrections are zero, find the second-order correction and comment. Approximately how small does V_0 have to be for the delta function to be considered a perturbation?