Department of Physics

Preliminary Exam January 3–7, 2000

Day 3: Quantum Mechanics and Modern Physics

Thursday, January 6, 2000

9:00 a.m. – 12:00 p.m.

Instructions:

1. Write the answer to each question on a separate sheet of paper. If more than one sheet is required, staple all the pages corresponding to a *single* question together in the correct order. But, do *not* staple all problems together. This exam has *six* questions.

2. Be sure to write your identification number (*not* your name!) and the problem number on each problem sheet.

3. The time allowed for this exam is three hours. All questions carry the same amount of credit. Manage your time carefully.

4. If a question has more than one part, it may not be necessary to successfully complete one part in order to do the other parts.

5. The exam will be evaluated, in part, by such things as the clarity and organization of your responses. It is a good idea to use short written explanatory statements between the lines of a derivation, for example. Be sure to substantiate any answer by calculations or arguments as appropriate. Be concise, explicit, and complete.

6. The use of electronic calculators is permitted. However, obtaining preprogrammed information from programmable calculators or using any other reference material is strictly prohibited. Oklahoma State University Policies and Procedures on Academic Dishonesty and Academic Misconduct will be followed.

Some physical constants

speed of light in vacuum	c	2.998×10^{8} m/s
Planck's constant	$h = 2\pi\hbar$	6.626×10^{-34} J · s
electron rest mass	m_e	9.109×10^{-31} kg
electron charge	$\lvert e \rvert$	1.602×10^{-19} C

Problem 1

Consider a potential well of the form, $V(x) \propto |x|$. (This potential confines a particle of mass m.)

(a) (Qualitative) *Sketch* the ground-state and first two excited-state wavefunctions for this potential, included with a sketch of the potential itself. (In your sketch pay attention to: symmetries; curvatures; number of nodes; asymptotic behaviors.)

(b) (Quantitative) *Estimate* the ground-state energy if the potential is given as $V(x) = V_0\,|x|/a$. (A semiclassical estimate will suffice.)

(c) (Qualitative) Do you expect the eigenenergy spacing to: *increase; decrease; remain constant;* with increasing eigenenergy?
(Hint: Side by side, sketch $V(x)$ vs x and corresponding eigenenergy levels for potential wells of the form, $V(x) \propto |x|^n$, for the cases, $n = 1$, 2, and ∞.)

(d) Finally, consider the potential wells $V_1(x) \propto |x|$ and

$$V_2(x) = \begin{cases} V_1(x), & x \geq 0, \\ +\infty, & x < 0. \end{cases}$$

Side by side, sketch these potentials. Now, sketch in energy levels and wavefunctions for the first three eigenstates of each potential, being careful to indicate any alignments in the energy spectra between the two cases.

Problem 2

(a) Obtain an expression for the De Broglie wavelength of a free electron with kinetic energy E_k, in the following cases:

i. non-relativistic limit

ii. extreme relativistic limit

iii. general case

(b) Sketch $1/\lambda$ vs E_k. Roughly, what magnitude of E_k marks the cross-over from non-relativistic to extreme-relativistic behavior?

(c) Numerically compute λ for an electron beam of energy

i. 5.00 keV

ii. 500. keV

iii. 50.0 MeV

Problem 3

(a) A spaceship begins its voyage to planet Omega 1.6 light years away from Earth. If it travels at a constant speed $v = 0.8c$ (c is the speed of light), what will be the elapsed time, as measured by an observer stationary on Earth, when the ship reaches Omega? What will be the elapsed time as measured by a traveler in the spaceship? How do you account for the difference (if any) in these measurements from the space-traveler's perspective?

(b) An electron collides head-on with a positron (which has the same rest energy of an electron) which is at rest. The two particles annihilate, producing two gamma rays (photons). If the energy of each gamma ray is measured to be 4 MeV:

i. What was the kinetic energy of the incident electron?

ii. What is the scattering angle—the angle each photon makes with the original direction of the incident electron?

Problem 4

A two-state system has energy eigenstates and eigenvalues defined by the following solutions to the time-independent Schrödinger equation:

$$\hat{H}|1\rangle = E_1|1\rangle, \qquad \hat{H}|2\rangle = E_2|2\rangle.$$

(a) What are the corresponding solutions, Ψ_1 and Ψ_2, to the time-dependent Schrödinger equation?

(b) Assume the system to be in a normalized superposition state $\Psi = a\Psi_1 + b\Psi_2$, where $\langle\Psi|\Psi\rangle = 1$. Find the expectation value $\langle\mu\rangle$ of the electric dipole moment $\hat{\mu}$ in this state, writing your result in terms of the matrix elements μ_{mn} $(m, n = 1, 2)$, where

$$\mu_{mn} \equiv \langle m|\hat{\mu}|n\rangle.$$

(To simplify your answer, assume that a, b, and μ_{mn} are real.)

(c) A semiclassical interpretation of the fact that $\langle\mu\rangle$ has a harmonically time-varying part is that the system in the state Ψ above is radiating an electromagnetic wave. What is the frequency of this wave? Which superposition of eigenstates yields the largest field amplitude?

(d) If an electromagnetic wave of frequency ω is incident on the system in state Ψ, then the interaction energy is

$$V = -\langle\mu\rangle E \cos\omega t \ ,$$

where E is the amplitude of the wave. Given that the strength of this interaction is determined by averaging V over a time long compared to the period of the wave, explain how the concept of resonance in the interaction results from this simple semiclassical picture.

Problem 5

A spin-1/2 particle is described by the Hamiltonian,

$$H = a(\sqrt{3}S_x + S_z),$$

where a is a constant and (S_x, S_y, S_z) are the three components of the spin operator:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) Determine the eigenvalues and eigenfunctions of H.

(b) What is the probability of measuring the spin to be $+\hbar/2$ along the z-direction for these two eigenstates?

(c) One linear combination of S_x, S_y, and S_z is a conserved quantity in this problem. Identify this combination. What additional spin observable is conserved here?

Problem 6

An atomic hydrogen source emits 10.2-eV photons that strike a metal whose work function is 4.6 eV. Some of the electrons emitted via the photoelectric effect pass through a selector that produces a nearly monoenergetic electron beam. This electron beam is then incident on a double slit, and interference fringes are detected on a screen behind the double slit.

(a) What is the wavelength of the photons, and what transition in hydrogen produces them?

(b) What is the maximum kinetic energy of the emitted photoelectrons?

(c) If the selected electrons incident on the double slit have a kinetic energy of 2.0 eV, what is their De Broglie wavelength?

(d) The selected electrons are not truly monoenergetic; their energies are distributed about a mean of 2.0 eV, and this distribution has a width of 0.2 eV. Assume that the double-slit widths are small compared to their separation. Explain why the number of fringes observed is limited, and estimate the number of fringes detected.