Department of Physics

Preliminary Exam January 5–9, 2004

Day 3: Quantum Mechanics and Modern Physics

Thursday, January 8, 2004

9:00 a.m. – 12:00 p.m.

Instructions:

1. Write the answer to each question on a separate sheet of paper. If more than one sheet is required, staple all the pages corresponding to a *single* question together in the correct order. But, do *not* staple all problems together. This exam has *five* questions.

2. Be sure to write your exam identification number (*not* your name or student ID number!) and the problem number on each problem sheet.

3. The time allowed for this exam is three hours. All questions carry the same amount of credit. Manage your time carefully.

4. If a question has more than one part, it may not be necessary to successfully complete one part in order to do the other parts.

5. The exam will be evaluated, in part, by such things as the clarity and organization of your responses. It is a good idea to use short written explanatory statements between the lines of a derivation, for example. Be sure to substantiate any answer by calculations or arguments as appropriate. Be concise, explicit, and complete.

6. The use of electronic calculators is permitted. However, obtaining preprogrammed information from programmable calculators or using any other reference material is strictly prohibited. Oklahoma State University Policies and Procedures on Academic Dishonesty and Academic Misconduct will be followed.

Some physical constants

speed of light in vacuum	c	2.998×10^8 m/s
Planck's constant	$h = 2\pi\hbar$	6.626×10^{-34} J·s
electron rest mass	m_e	9.109×10^{-31} kg
electron charge	$\lvert e \rvert$	1.602×10^{-19} C

Problem 1

(a) Discuss the process(es) underlying Compton scattering, that is, the effect in which a photon loses energy after interacting with some material.

(b) By considering momentum and energy relativistically show that the change in wavelength of the scattered photon is:

$$\Delta\lambda = \frac{h}{m_e c}(1 - \cos\theta)$$

where m_e is the mass of an electron and θ is the angle through which the photon is scattered.

(c) If the wavelength of scattered photons is shifted by 1% when the scattering angle is 60 degrees, what is the wavelength of the incident photons? Why would it be desirable to use short wavelength radiation in this type of experiment?

Problem 2

An "angular momentum" operator can be defined as any Hermitian vector operator $\mathbf{J}$ satisfying $\mathbf{J}\times\mathbf{J} = i\hbar\,\mathbf{J}$ (i.e. $[J_\alpha, J_\beta] = i\hbar\,\varepsilon_{\alpha\beta\gamma}J_\gamma$).

(a) Using the above relationship, show that $[J_z, J_\pm] = \pm\hbar\,J_\pm$, where $J_\pm = J_x \pm i\,J_y$.

(b) Also show explicitly for the x, y, and z Cartesian components of $\mathbf{L} = \mathbf{r}\times\mathbf{p}$, where $\mathbf{p} = -i\hbar\nabla$, that $\mathbf{L}\times\mathbf{L} = i\hbar\,\mathbf{L}$.

(c) Show that $[J^2, J_\alpha] = 0$, for α any one of the Cartesian components x, y, or z.

Problem 3

Consider the one-dimensional model problem of a particle with mass m and total energy E in the presence of a semi-infinite step potential (see figure)

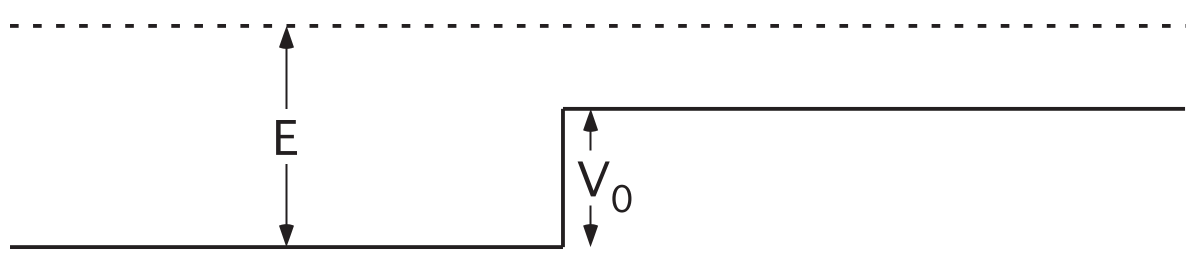
$$V(x) = \begin{cases} V_0, & x > 0, \\ 0, & x < 0, \end{cases}$$

where $V_0 > 0$.

(a) Write the general solution for the time-independent Schrödinger's equation for this system with $E > V_0$.

(b) Find the solution that corresponds to the situation of a particle incident from the left with a finite probability of either being transmitted into the right-hand region or reflected back into the left-hand region.

(c) Work out the expression that gives the probability, R, the incident particle will be reflected.



Problem 4 (*See the Useful Information below*)

Two spin $\frac{1}{2}$ particles (e.g. electrons) interact via an "exchange" mechanism such that

$$\hat{H}_0 = J_{12}\hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2.$$

(a) Show that

$$\hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2 = \tfrac{1}{2}[\hat{s}_{+1}\hat{s}_{-2} + \hat{s}_{-1}\hat{s}_{+2}] + \hat{s}_{z1}\hat{s}_{z2}.$$

(b) Find (and discuss) the eigenvalues and eigenvectors of $\hat{H}_0$.

(c) An external magnetic field B_0 is imposed in the z direction and there is an additional interaction

$$\hat{H}_1 = -[\{-\mu_0/\hbar\}(\hat{s}_{z1} + \hat{s}_{z2})]B_0 = \{\mu_0/\hbar\}B_0(\hat{s}_{z1} + \hat{s}_{z2}) = A(\hat{s}_{z1} + \hat{s}_{z2}).$$

Find the eigenvectors and eigenvalues of $\hat{H} = \hat{H}_0 + \hat{H}_1$.

(d) If the system is prepared so that at $t = 0$ the state vector is

$$|\Psi(0)\rangle = \tfrac{1}{2}\big\{|\alpha\rangle_1|\alpha\rangle_2 + |\alpha\rangle_1|\beta\rangle_2 + |\beta\rangle_1|\alpha\rangle_2 + |\beta\rangle_1|\beta\rangle_2\big\},$$

what is the probability of finding the system in state $|\alpha\rangle_1|\beta\rangle_2$ at the later time t?

Useful Information:

The vector operator $\hat{\vec{s}}_i$ $(i = 1, 2)$ has Cartesian components $\hat{s}_{xi}$, $\hat{s}_{yi}$, and $\hat{s}_{zi}$. Three equivalent operators are $\hat{s}_{+i}$, $\hat{s}_{-i}$, and $\hat{s}_{zi}$ for particle $i = 1, 2$, where these so-called raising and lowering operators are defined as $\hat{s}_{+i} = \hat{s}_{xi} + i\hat{s}_{yi}$ and $\hat{s}_{-i} = \hat{s}_{xi} - i\hat{s}_{yi}$.

The respective single-particle states are

$$|\alpha\rangle \equiv |s = \tfrac{1}{2}; m_s = \tfrac{1}{2}\rangle \quad \text{and} \quad |\beta\rangle \equiv |s = \tfrac{1}{2}; m_s = -\tfrac{1}{2}\rangle,$$

for which

$$\hat{s}^2|\alpha\rangle = \tfrac{1}{2}(\tfrac{1}{2}+1)\hbar^2|\alpha\rangle = \tfrac{3}{4}\hbar^2|\alpha\rangle \quad \text{and} \quad \hat{s}_z|\alpha\rangle = \tfrac{1}{2}\hbar|\alpha\rangle,$$
$$\hat{s}^2|\beta\rangle = \tfrac{1}{2}(\tfrac{1}{2}+1)\hbar^2|\beta\rangle = \tfrac{3}{4}\hbar^2|\beta\rangle \quad \text{and} \quad \hat{s}_z|\beta\rangle = -\tfrac{1}{2}\hbar|\beta\rangle,$$
$$\hat{s}_+|\alpha\rangle = 0, \qquad \hat{s}_-|\alpha\rangle = \hbar|\beta\rangle,$$
$$\hat{s}_+|\beta\rangle = \hbar|\alpha\rangle, \quad \hat{s}_-|\beta\rangle = 0.$$

A basis for the two-particle states is thus

$$|\alpha\alpha\rangle = |\alpha\rangle_1 \otimes |\alpha\rangle_2 = |\alpha\rangle_1\,|\alpha\rangle_2,$$
$$|\alpha\beta\rangle = |\alpha\rangle_1 \otimes |\beta\rangle_2 = |\alpha\rangle_1\,|\beta\rangle_2,$$
$$|\beta\alpha\rangle = |\beta\rangle_1 \otimes |\alpha\rangle_2 = |\beta\rangle_1\,|\alpha\rangle_2,$$
$$|\beta\beta\rangle = |\beta\rangle_1 \otimes |\beta\rangle_2 = |\beta\rangle_1\,|\beta\rangle_2.$$

Problem 5

Consider a one-dimensional potential energy

$$V(x) = \begin{cases} \infty, & -\infty \leq x < 0, \\ \frac{V_0}{a}x, & 0 \leq x \leq \infty. \end{cases}$$

Use the variational principle to obtain an upper bound on the ground state energy of a particle of mass m in the potential energy. In particular assume trial functions

$$\psi(x) = N x^n e^{-\beta x/2}, \quad n \geq 1$$

and show that:

$$\langle \hat{H} \rangle = \frac{1}{4(2n-1)}\,\frac{\hbar^2\beta^2}{2m} + (2n+1)\,\frac{V_0}{\beta a}.$$

Find the optimal upper bound to the ground state energy based on this result.

Note that:

$$\int_0^\infty x^p e^{-ax} dx = \frac{p!}{a^{p+1}}.$$