Department of Physics

Preliminary Exam January 3–6, 2006

Day 1: Classical Mechanics

Tuesday, January 3, 2006
### 9:00 a.m.–12:00 p.m.

Instructions:

1. Write the answer to each question on a separate sheet of paper. If more than one sheet is required, staple all the pages corresponding to a *single* question together in the correct order. But, do *not* staple all problems together. This exam has *five* questions.

2. Be sure to write your exam identification number (*not* your name or student ID number!) and the problem number on each problem sheet.

3. The time allowed for this exam is three hours. All questions carry the same amount of credit. Manage your time carefully.

4. If a question has more than one part, it may not always be necessary to successfully complete one part in order to do the other parts.

5. The exam will be evaluated, in part, by such things as the clarity and organization of your responses. It is a good idea to use short written explanatory statements between the lines of a derivation, for example. Be sure to substantiate any answer by calculations or arguments as appropriate. Be concise, explicit, and complete.

6. The use of electronic calculators is permissible and may be needed for some problems. However, obtaining preprogrammed information from programmable calculators or using any other reference material is strictly prohibited. The Oklahoma State University Policies and Procedures on Academic Dishonesty and Academic Misconduct will be followed.

Problem 1

A disk is mounted on a shaft through the center of mass of a block which slides down a frictionless plane incline at an angle θ as shown in the figure. The radius of the disk is a and the axle is offset a distance b from the center of the disk. The mass of the disk is m and its moment of inertia about an axis through the center of mass is I. The mass of the block is M.
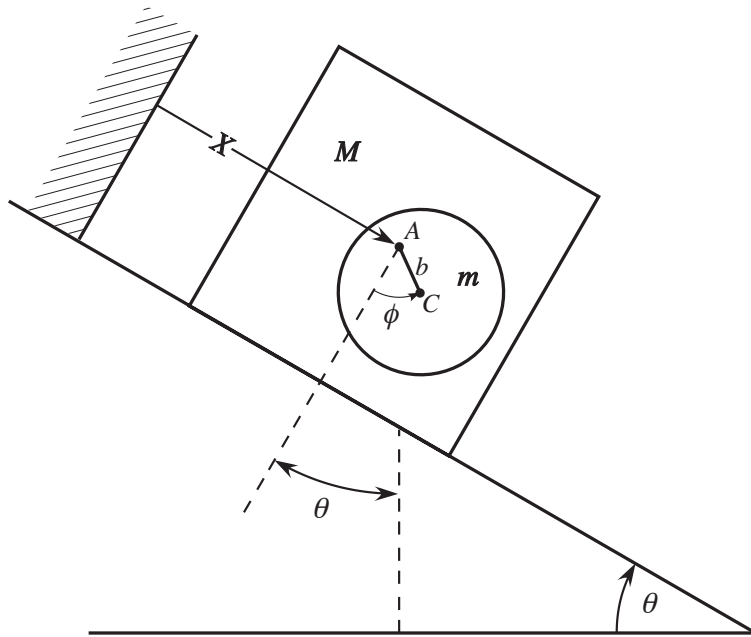
> The axle A passes through the center of mass of the block of mass M. The disk has a mass m. The disk is pivoted on an axle through A at a distance b from the center of the disk C. I is the moment of inertia about C. The generalized coordinates are X and ϕ; ϕ is measured from the line through A which is perpendicular to the plane. θ is the angle made by the incline with the horizontal.

See the figure and note the definition of the angle ϕ which is measured from the line (plane) perpendicular to the incline.

(a) Write down the Lagrangian for the system in terms of the generalized coordinates X (the center of mass of the block) and ϕ.

(b) Show that the equations of motion are

$$(M + m)\ddot{X} + mb\ddot{\phi}\cos\phi - mb\dot{\phi}^2\sin\phi - (M + m)g\sin\theta = 0$$

and

$$mb\ddot{X}\cos\phi + mb^2\ddot{\phi} + I\ddot{\phi} + mgb\sin(\phi - \theta) = 0$$

(c) Show that $\phi(t) \equiv 0$ (for all time) is a solution. Why does the disk "hang" at an angle with respect to the vertical? Interpret this result in terms of the non-inertial frame defined by the sliding block.

(d) Show that the equation of motion for "the disk" can be written as

$$(mb^2 + I)\ddot{\phi} - \frac{m^2b^2(\ddot{\phi}\cos^2\phi - \dot{\phi}^2\sin\phi\cos\phi)}{(M + m)} + (mgb\cos\theta)\sin\phi = 0$$

(e) Find the solution to part (d) for the case where $M \gg m$ and the amplitude of the motion is small, $|\phi| \ll 1$.

(f) In the special case where $\theta = 0$ (and again with $M \gg m$) and initial conditions $\phi = \phi_0$, $\dot{\phi} = 0$, show that once the disk is released, the disk swings and the block rocks.

Problem 2

The European Space Agency is now accepting European contractors' bids for the "Don Quijote" space mission that will test the collision technique for altering asteroid orbits. The mission might be launched as early as 2011. This test is part of a larger effort to develop effective deflection methods for protecting Earth from the impacts of large (> 1 km) asteroids.
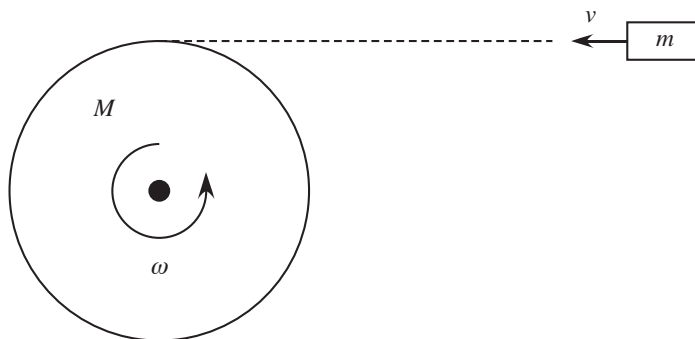
(a) The target asteroid will have a diameter of about 500 m, and a mean density near 2000 kg/m³. Use those numbers to compute the asteroid's mass M, assuming that it is spherical.

(b) The interceptor spacecraft will have a mass m of 400 kg, and hit the asteroid with a relative speed v of 15 km/s. A companion spacecraft will measure how the impact affects the asteroid's velocity. Assuming that the collision is perfectly inelastic, compute the magnitude of the asteroid's velocity change Δv in the limit $M \gg m$.

(c) The companion spacecraft will additionally measure how the impact affects the asteroid's rotation. As a first step in evaluating rotational effects, show that the moment of inertia I of a uniformly dense sphere of mass M and radius R rotating about any diameter is

$$I = \frac{2}{5}MR^2$$

(d) Suppose that the target asteroid is spherical, uniformly dense, and spinning with a typical rotational period of 6 hours on an axis through its center. Compute its angular frequency ω in rad/s.

(e) Calculate the magnitude of the asteroid's angular frequency change $\Delta\omega$ of the asteroid after impact in the limit $M \gg m$. Assume that the asteroid's moment of inertia remains unchanged, and that the collision occurs tangentially, in the equatorial plane of the asteroid, and in its direction of rotation.

Problem 3

Three point particles of equal mass m are connected by three massless springs obeying Hooke's law and having spring constants $4k$, $7k$ and $11k$ respectively. All three springs have equal unstressed lengths l. The spring with spring constant $4k$ is attached to the ceiling, as shown in the figure. The system is disturbed so that the masses oscillate in a vertical straight line.
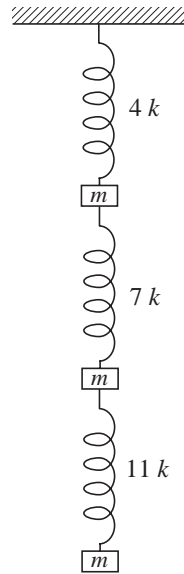
(a) Determine the equilibrium configuration of the system. That is, determine the positions of the three masses in equilibrium.

(b) Construct the Lagrangian for the system.

(c) Obtain the normal modes and the normal frequencies for small oscillation in the vertical direction. You may leave the normalization of the normal modes arbitrary. (Hint: Look for factoring of the eigenvalue equation.)

(d) Sketch qualitatively the relative amplitudes of the normal modes.

Problem 4

The equation for the radial motion of a mass in a central potential is

$$m\ddot{r} = -\frac{\partial}{\partial r}\left(V(r) + \frac{\ell^2}{2mr^2}\right) \equiv -\frac{\partial}{\partial r}U(r)$$

where

$$U(r) = V(r) + \frac{\ell^2}{2mr^2}$$

and the constant of motion ℓ is defined by

$$mr^2\dot{\phi} = \ell$$

(a) Show that a second constant of motion is

$$E = \frac{1}{2}m\dot{r}^2 + U(r)$$

(b) Assume $V(r) = \frac{1}{2}kr^2$. Then, show that, for a given ℓ, the radius of the circular orbit, the energy, and the angular velocity are given by

$$r_0 = (\ell^2/mk)^{1/4}, \qquad E = (k\ell^2/m)^{1/2}, \qquad \dot{\phi} = \frac{\ell}{mr_0^2} = (k/m)^{1/2} \equiv \omega.$$

(c) Further assume that this circular orbit is suddenly perturbed by a radial impulse. Determine (approximately) the perturbed orbit, showing that

$$r(t) \simeq r_0 + A\cos 2\omega t$$

and

$$\phi(t) \simeq \omega t - \frac{A}{r_0}\sin 2\omega t$$

where $A \ll r_0$ is the amplitude of the radial displacement from the circular orbit.

Problem 5

A uniform bar of mass M and length $2l$ is suspended from one end by a massless spring of force constant k and equilibrium length d. The bar can swing freely only in one vertical plane, and the spring is constrained to move only in the vertical direction.

(a) Set up the Lagrangian for the system.

(b) Find the Hamiltonian for the system and obtain the Hamilton's equations of motion.

(c) Are there any ignorable coordinates? What quantities are conserved in the problem?