

Department of Physics
Preliminary Exam January 3–7, 2011
Day 3: Quantum Mechanics and Modern Physics
Thursday, January 6, 2011
9:00 a.m. – 12:00 p.m.

Instructions:

1. Write the answer to each question on a separate sheet of paper. If more than one sheet is required, staple all the pages corresponding to a *single* question together in the correct order. But, do *not* staple all problems together. This exam has *five* questions.
2. Be sure to write your exam identification number (*not* your name or student ID number!) and the problem number on each problem sheet.
3. The time allowed for this exam is three hours. All questions carry the same amount of credit. Manage your time carefully.
4. If a question has more than one part, it may not always be necessary to successfully complete one part in order to do the other parts.
5. The exam will be evaluated, in part, by such things as the clarity and organization of your responses. It is a good idea to use short written explanatory statements between the lines of a derivation, for example. Be sure to substantiate any answer by calculations or arguments as appropriate. Be concise, explicit, and complete.
6. The use of electronic calculators is permissible and may be needed for some problems. However, obtaining preprogrammed information from programmable calculators or using any other reference material is strictly prohibited. The Oklahoma State University Policies and Procedures on Academic Integrity will be followed.

There are five problems. Answer all five. Each problem carries 20 points. Include all relevant intermediate steps.

You may need these fundamental constants:

$$h = 2\pi\hbar = 6.63 \times 10^{-34} \text{ J s} = 4.14 \times 10^{-21} \text{ MeV s}$$

$$c = 2.99 \times 10^8 \text{ m/s}$$

$$e = 1.60 \times 10^{-19} \text{ C}$$

$$\text{Ry} = 13.6 \text{ eV}$$

$$m_e = 9.11 \times 10^{-31} \text{ kg} = 0.511 \text{ MeV}/c^2$$

$$m_p = 1.67 \times 10^{-27} \text{ kg} = 938.27 \text{ MeV}/c^2$$

Problem 1

- (a) First, write down the expression for the total relativistic energy, E , of an electron in terms of its rest mass, m_e , its momentum, $|\vec{p}|$, and the speed of light, c .
- (b) A high-energy photon with frequency ν is scattered, after colliding with an electron initially at rest in the laboratory reference frame, through an angle of 90° , as measured in the laboratory frame. Using the above relativistic expression for the electron energy, as well as conservation of total energy and momentum, derive an expression for the ratio, $\frac{\nu'}{\nu}$ of the final photon frequency, ν' , to the initial photon frequency. This ratio should be expressed in terms of m_e , c , ν , and the Planck's constant, h . Evaluate $\frac{\nu'}{\nu}$ if the incident photon energy is 1 MeV.

Problem 2

The Hamiltonian H for a system is given in a certain basis as

$$H = \epsilon_0 \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 2i \\ 0 & -2i & 0 \end{pmatrix} .$$

In the same basis, an operator A corresponding to a certain observable has the matrix form

$$A = a_0 \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} .$$

- (a) If the energy of the system is measured, what are the possible outcomes?
- (b) If the system is initially in a state given by

$$|\psi\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} ,$$

what is the probability that an energy measurement will yield $\sqrt{5}\epsilon_0$?

What is the state of the system immediately after such a measurement?

What are the probabilities for energy measurements other than $\sqrt{5}\epsilon_0$?

- (c) What would be the expectation value of A , if the measurement is done immediately after measuring the energy of the system to be $\sqrt{5}\epsilon_0$?

What would be the variance ΔA in the measurement?

Problem 3

Consider a particle of mass m in a one-dimensional infinite potential well with a thin wall at the center of the well. The potential energy is given as

$$V(x) = \begin{cases} \infty, & |x| > a \\ V_0 \delta(x), & |x| < a \end{cases}$$

where V_0 is a positive constant.

- (a) Obtain the wave function of the particle inside the well by solving the time-independent Schrödinger equation and applying the appropriate boundary conditions. You need not normalize the wave function.
 - (b) Write down the quantization condition for the momentum of the particle. Qualitatively describe the structure of energy levels and the corresponding wave functions and compare them to the case without the central wall (i.e., the case where $V_0 = 0$).
 - (c) Determine the relative splitting between the energies of the lowest two levels, in the case of a large barrier parameter $V_0 \gg \frac{\hbar^2}{ma}$.
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Problem 4

- (a) In a muonic Hydrogen atom, the electron of an ordinary Hydrogen atom is replaced by a muon, which is 207 times heavier than the electron. Find the principal quantum number n such that the wavelength corresponding to the $n+1$ to n transition in the muonic Hydrogen atom is the closest to the wavelength for transition between the two lowest lying energy levels of an ordinary Hydrogen atom.
- (b) One of the atoms in the (ordinary) H_2 molecule captures a negative muon into the $n = 3$ state. The subsequent $n = 3$ to $n = 2$ transition produces emission of an X-ray and causes the molecule to recoil. Assuming that the energy lost to vibrational modes in the molecule is negligible, *estimate* the Doppler broadening that results from this muon level transition. You may assume for this estimate that the energy levels for the muon in H_2 molecule are approximately the same as the levels in the muonic Hydrogen atom.

Problem 5

The ground state wave function for a one-dimensional harmonic oscillator ($H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2$) is

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/(2\hbar)}$$

- (a) Verify that $\psi_0(x)$ is a solution to the time-independent Schrödinger equation.
- (b) Sketch qualitatively the probability density $|\psi_0(x)|^2$ as a function of x , and indicate on the plot the classically forbidden regions for the oscillator. Explain why the quantum oscillator can move into the classically forbidden region. Sketch also (qualitatively) the probability density of the oscillator in the first excited state.
- (c) Consider now adding a small perturbation to the Hamiltonian given by

$$H_1 = \beta X^4$$

Assuming β to be small, compute the first order (in β) shift in the energy E_n of the system in the state $|n\rangle$.

Suggestion: For part (c), you may wish to work in the energy basis. Some useful information:

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle,$$

where

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(X + \frac{iP}{m\omega} \right).$$