Department of Physics

Preliminary Exam January 2–5, 2013

Day 3: Quantum Mechanics and Modern Physics

Friday, January 4, 2013

9:00 a.m.–12:00 p.m.

Instructions:

1. Write the answer to each question on a separate sheet of paper. If more than one sheet is required, staple all the pages corresponding to a *single* question together in the correct order. But, do *not* staple all problems together. This exam has *five* questions.

2. Be sure to write your exam identification number (*not* your name or student ID number!) and the problem number on each problem sheet.

3. The time allowed for this exam is three hours. All questions carry the same amount of credit. Manage your time carefully.

4. If a question has more than one part, it may not always be necessary to successfully complete one part in order to do the other parts.

5. The exam will be evaluated, in part, by such things as the clarity and organization of your responses. It is a good idea to use short written explanatory statements between the lines of a derivation, for example. Be sure to substantiate any answer by calculations or arguments as appropriate. Be concise, explicit, and complete.

6. The use of electronic calculators is permissible and may be needed for some problems. However, obtaining preprogrammed information from programmable calculators or using any other reference material is strictly prohibited. The Oklahoma State University Policies and Procedures on Academic Integrity will be followed.

Some useful formulas:

$$
\begin{aligned}
\sin^2 x &= \frac{1-\cos(2x)}{2} \\
\cos^2 x &= \frac{1+\cos(2x)}{2} \\
\cos x \cos y &= \frac{\cos(x-y)+\cos(x+y)}{2} \\
\sin x \sin y &= \frac{\cos(x-y)-\cos(x+y)}{2} \\
\sin x \cos y &= \frac{\sin(x+y)+\sin(x-y)}{2} \\
\cos x \sin y &= \frac{\sin(x+y)-\sin(x-y)}{2}
\end{aligned}
$$

Problem 1

Consider a particle of mass m moving freely in a one-dimensional infinite potential well stretching from $x = 0$ to $x = L$. The potential energy is

$$V(x) = \begin{cases} 0, & 0 \le x \le L \\ \infty, & x > L \text{ or } x < 0\,. \end{cases}$$

(a) Obtain the allowed energy eigenvalues and the corresponding eigenfunctions.

(b) Sketch the wavefunction in the $n = 3$ state.

(c) Show that the wavefunctions corresponding to different energy levels are orthogonal.

(d) Now, the size of the well is suddenly quadrupled (i.e., the right–hand side of the wall is moved from $x = L$ to $x = 4L$). If the particle was initially in the ground state, calculate the probability of finding it in the ground state of the new well.

Problem 2

A hydrogen atom is prepared in a linear superposition of the $1s$ and $2p$ states (spin is neglected here). The state of the atom may be written in terms of wavefunctions ψ_{nlm_l} as

$$\psi = \frac{1}{\sqrt{2}}\psi_{100} + \frac{e^{i\pi/4}}{\sqrt{2}}\psi_{211}.$$

(a) If the energy is measured, what results may be found? What is the expectation value of the energy in the state ψ? Give your answers in eV.

(b) If the square of the orbital angular momentum (L^2) is measured, what results may be found? What is the expectation value of the square of the orbital angular momentum in the state ψ?

(c) If the z-component of the orbital angular momentum (L_z) is measured, what results may be found? What is the expectation value of the z-component of the orbital angular momentum in the state ψ?

(d) Suppose that, after an energy measurement, the atom is found in the $1s$ state. Its normalized wave function is then given by

$$\psi_{100}(r, \theta, \phi) = \frac{1}{\sqrt{\pi} a_0^{3/2}} e^{-r/a_0}.$$

Find the expectation value of $1/r^2$ in this state.

Problem 3

A particle of mass m is placed in a potential well of width a:

$$V(x) = \begin{cases} \infty, & x < 0 \\ 0, & 0 < x < a \\ V_0, & x > a \end{cases} .$$

(a) Write down the ground-state wave function in each of the regions. Do not derive the normalization coefficients.

(b) Write down the boundary conditions and show how they lead to quantization of energy.

(c) The energy of the ground state is $E_1 = V_0/2$. Obtain the width of the well in terms of V_0 and m.

Problem 4

The Hamiltonian of a spin-1/2 particle is given by

$$H = a\,\sigma_z + b\,\sigma_x$$

where σ_i are the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) What are the possible outcomes if the energy of the particle is measured?

(b) Obtain the energy eigenfunctions for the system.

(c) What is the probability that a measurement of spin will result in $\sigma_z = -1$, if initially the particle is known to be in one of the energy eigenstates?

(d) Are any of the spin components σ_i conserved? What (other) spin observable is conserved?

Problem 5

On their 20^{th} birthday, twins A and B are separated when B embarks on a trip to a planetary system 24 light-years away. Assume that B travels at a constant speed of $v = 0.8c$ relative to A back on Earth, and turns around immediately after arriving, to return to Earth.

(a) When B returns and the twins are reunited, what are their ages?

(b) Confirm your answer to part (a) by considering the following: every time a year passes for A, he sends a birthday greeting to B (at the speed of light); likewise, every time a year passes for B, she sends a birthday greeting to A. The frequency of this signal (one per year in the sender's frame), is seen Doppler shifted by the receiver because of their relative motion. Show that each twin receives the number of birthday greetings that corresponds to the number of years that have passed for the other twin. Recall that the relativistic Doppler shift is given by

$$\nu' = \nu \sqrt{\frac{1 \pm v/c}{1 \mp v/c}}\,.$$