Department of Physics

Preliminary Exam: January 6–10, 2014

Day 4: Thermodynamics and Statistical Physics

Friday, January 10, 2014

9:00 a.m.–12:00 p.m.

Instructions:

1. Write the answer to each question on a separate sheet of paper. If more than one sheet is required, staple all the pages corresponding to a *single* question together in the correct order. But, do *not* staple all problems together. This exam has *five* questions.

2. Be sure to write your exam identification number (*not* your name or student ID number!) and the problem number on each problem sheet.

3. The time allowed for this exam is three hours. All questions carry the same amount of credit. Manage your time carefully.

4. If a question has more than one part, it may not always be necessary to successfully complete one part in order to do the other parts.

5. The exam will be evaluated, in part, by such things as the clarity and organization of your responses. It is a good idea to use short written explanatory statements between the lines of a derivation, for example. Be sure to substantiate any answer by calculations or arguments as appropriate. Be concise, explicit, and complete.

6. The use of electronic calculators is permissible and may be needed for some problems. No other electronic device is permitted. Obtaining preprogrammed information from programmable calculators or using any other reference material is strictly prohibited. The Oklahoma State University Policies and Procedures on Academic Integrity will be followed.

Attempt **all five** *problems. Each graded problem carries 20 points.*

Boltzmann's constant:	$k_B = 1.38 \times 10^{-23}\,\mathrm{J/K}$
Gas constant:	$R = N_A k_B = 8.31\,\mathrm{J/(mol{\cdot}K)}$
Unified atomic mass unit:	$u = 1.66 \times 10^{-27}\,\mathrm{kg}$

Gaussian integrals:

$$\int_0^\infty \exp(-\lambda x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}\,, \qquad \int_0^\infty x^3 \exp(-\lambda x^2)\,dx = \frac{1}{2\lambda^2}\,,$$

$$\int_0^\infty x \exp(-\lambda x^2)\,dx = \frac{1}{2\lambda}\,, \qquad \int_0^\infty x^4 \exp(-\lambda x^2)\,dx = \frac{3}{8}\sqrt{\frac{\pi}{\lambda^5}}\,,$$

$$\int_0^\infty x^2 \exp(-\lambda x^2)\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\lambda^3}}\,, \qquad \int_0^\infty x^5 \exp(-\lambda x^2)\,dx = \frac{1}{\lambda^3}\,,$$

$$\int_0^\infty x^n \mathrm{e}^{-ax}\,dx = \frac{\Gamma(n+1)}{a^{n+1}}\,, \qquad \Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(x+1) = x\Gamma(x).$$

Problem 1

One mole of a monatomic ideal gas is taken from an initial state of pressure P_0 and volume V_0 to a final state of pressure $2P_0$ and volume $2V_0$ via two different processes:

I. It expands isothermally until its volume is doubled, and then its pressure is increased at constant volume to the final state.

II. It is compressed isothermally until its pressure is doubled, and then its volume is increased at constant pressure to the final state.

(a) Show the path of each process on a P–V diagram.

(b) For each process, calculate the heat absorbed by the gas in terms of P_0 and V_0.

(c) For each process, calculate the work done on the gas in terms of P_0 and V_0.

(d) For each process, calculate the change in internal energy of the gas in terms of P_0 and V_0.

(e) For each process, calculate the change in entropy of the gas in terms of P_0 and V_0.

Problem 2

Consider N particles obeying classical statistics that have energies distributed in two levels ε_0 and $\varepsilon_0 + \delta$ ($\delta > 0$).

(a) Determine the partition function and free energy for the system.

(b) Determine the entropy.

(c) Determine the specific heat as a function of temperature, $C(T)$.

(d) Find the leading low-temperature and high-temperature behaviors of the specific heat and sketch a C–T plot.

Problem 3

Air in the middle levels of the atmosphere falls to the earth surface because of evaporative cooling, which can be understood in the following model. Consider an ideal gas system consisting of N_0 monatomic particles, each having mass m. The gas is contained in a cubic box of side L *and* the temperature is T.

(a) What is the normalized speed distribution of the gas? Derive the expression for the most probable speed.

(b) Derive the normalized distribution of kinetic energy. What is the most probable energy per particle?

(c) What is the average energy per particle?

(d) Instantaneously remove all particles from the system that have kinetic energy higher than $nk_{\mathrm{B}}T$ (n is an arbitrary positive real number). How does the temperature change? Qualitative explanation is sufficient.

Problem 4

Calculate the cost of producing one ton (1000 kg) of ice at $-10°C$ starting with water at room temperature (20°C) and assuming that the cost of electricity is 12 cents per kWh. Solve this problem by answering the following questions using the data below:

(a) How much energy must be extracted from the water in cal and in kWh? What dollar amount would that represent at \$0.12 per kWh?

(b) What would the cost be using a perfect reversible refrigerator?

Data: Specific heat of water = 1.00 cal/gram·°C
Specific heat of ice = 0.500 cal/gram·°C (1 cal = 4.186 J)
Latent heat of fusion (melting) = 80.0 cal/gram

Problem 5

A large number $\tilde{N}$ of spinless noninteracting bosons is confined to a cubic box of macroscopic size $V = L^3$ (with boundary x, y, or $z = 0$ or L) and in equilibrium at absolute temperature T.

(a) Write (without giving a detailed derivation) a formula for the complete set of solutions of the one-particle Schrödinger equation with nodes at the boundary that have energies

$$\varepsilon = \frac{\hbar^2 \pi^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2).$$

What are the allowed values of $\mathbf{n} = (n_x, n_y, n_z)$?

(b) Expand the grand canonical partition function $Z_{\text{gr}}(\beta, \mu, V)$ (with μ the chemical potential and $\beta = 1/k_{\text{B}}T$), as a product over $\mathbf{n}$, simplifying the expression for the factors within the product by performing the sum over the occupation number of each one-particle state $\mathbf{n}$.

(c) For very large V, we can replace the sum over $\mathbf{n}$ by an integral. Show that this leads to the density of states

$$D(\varepsilon) = \frac{V}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}.$$

(d) Apply this to $\dfrac{PV}{k_{\text{B}}T} = \log Z_{\text{gr}}$, while treating the term $\mathbf{n} = (1, 1, 1)$ separately.

(e) Derive from the previous result an expression for $\langle \tilde{N} \rangle$.

(f) What are the physical consequences related to the term $\mathbf{n} = (1, 1, 1)$?